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Theory and modelling of turbulent transport in astrophysical phenomena

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Topics

I. Turbulent transport

II. Turbulence modeling based on statistical theory

III. Global flow generation

IV. Stellar convection zone

V. Summary

I. Turbulent transport

Equation of fluctuating velocity $\mathbf{u} = \mathbf{U} + \mathbf{u}', \ \mathbf{U} = \langle \mathbf{u} \rangle, \ \mathbf{u}' = \mathbf{u} - \langle \mathbf{u} \rangle$

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial U_\alpha}{\partial x_a} - u'_a \frac{\partial u'_\alpha}{\partial x_a} + \frac{\partial}{\partial x_a} \langle u'_a u'_\alpha \rangle - \frac{\partial p'}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

turbulence–mean velocity interaction

turbulence–turbulence interaction

→ Instability or wave approach

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial U_\alpha}{\partial x_a} - \frac{\partial p'^{(\mathrm{R})}}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

Linear in $\mathbf{u}'$ and $p'^{(\mathrm{R})}$, each (Fourier) mode evolves independently

→ Closure approach

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial u'_\alpha}{\partial x_a} + \frac{\partial}{\partial x_a} \langle u'_a u'_\alpha \rangle - \frac{\partial p'^{(\mathrm{S})}}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

Homogeneous turbulence, no dependence on large-scale inhomogeneity

Nonlinear terms appear under the divergence operator

$$\nabla \cdot \mathbf{uu} \qquad (\mathbf{u} \cdot \nabla)\, \mathbf{u}$$

Integrated over the volume of the system → No net contribution only transfer

Fourier representations

$$\hat{f}(\mathbf{k};t) = \frac{1}{(2\pi)^3} \int f(\mathbf{r};t) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r} \qquad f(\mathbf{r};t) = \int \hat{f}(\mathbf{r};t) \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k}$$

$$\frac{1}{(2\pi)^3} \int f(\mathbf{r};t) g(\mathbf{r};t') \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}$$

$$= \frac{1}{(2\pi)^3} \int \left\{ \int \hat{f}(\mathbf{p};t) \mathrm{e}^{-i\mathbf{p}\cdot\mathbf{r}} d\mathbf{p} \right\} \left\{ \int \hat{g}(\mathbf{q};t') \mathrm{e}^{-i\mathbf{q}\cdot\mathbf{r}} d\mathbf{q} \right\} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}$$

$$= \frac{1}{(2\pi)^3} \int d\mathbf{r} \mathrm{e}^{+i(\mathbf{k}-\mathbf{q}-\mathbf{p})\cdot\mathbf{r}} \iint d\mathbf{p} d\mathbf{q} \hat{f}(\mathbf{p};t) \hat{g}(\mathbf{q};t')$$

$$= \iint d\mathbf{p} d\mathbf{q} \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \hat{f}(\mathbf{p};t) \hat{g}(\mathbf{q};t') \qquad \left[\delta(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d\mathbf{k}\ \mathrm{e}^{\pm i\mathbf{k}\cdot\mathbf{r}} \right]$$

Nonlinear term

$$\left[\widehat{\nabla \cdot \mathbf{uu}} \right]_\alpha = -ik_a \iint d\mathbf{p} d\mathbf{q}\ \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \hat{u}_a(\mathbf{p};t) \hat{u}_\alpha(\mathbf{q};t')$$

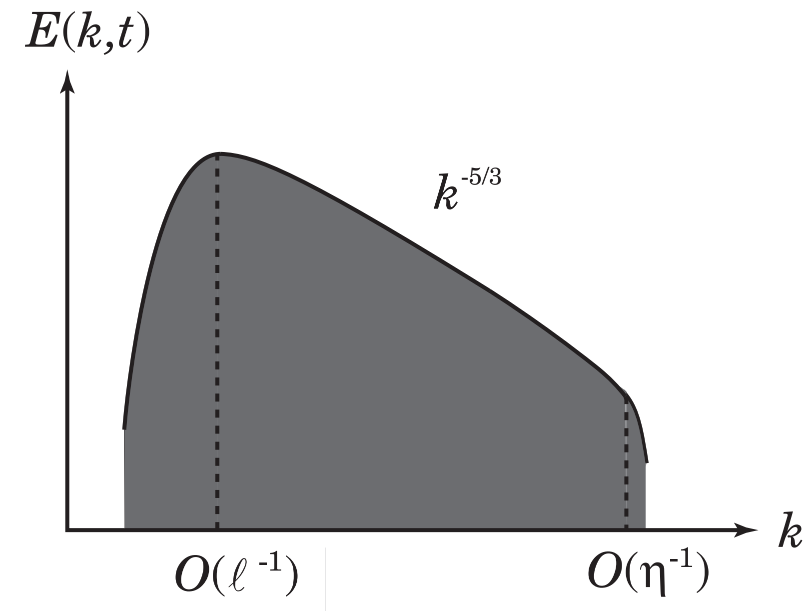
The dynamics of **k** mode is governed by its interaction with all other modes

Integral scale $\ell = \ell\{K, \varepsilon\}$

$$\ell \sim \frac{u^3}{\varepsilon} \sim \frac{K^{3/2}}{\varepsilon}$$

Kolmogorov microscale $\eta = \eta\{\nu, \varepsilon\}$

$$\eta \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$



$$\frac{\ell}{\eta} \sim \left(\frac{u\ell}{\nu}\right)^{3/4} = O\left(Re^{3/4}\right)$$

Required grid points $N_{\mathrm{G}} = \left(\frac{\ell}{\eta}\right)^3 = O\left(Re^{9/4}\right)$

	Re	*N*G
Walking	$O(10^4)$	$O(10^9)$
Cars	$O(10^6)$	$O(10^{13.5})$
Airplanes	$O(10^8)$	$O(10^{18})$

	Re	*N*G
Earth's outer core	$O(10^8)$	$O(10^{18})$
Solar convection zone	$O(10^{10})$	$O(10^{22.5})$
Galaxies	$O(10^{11})$	$O(10^{25})$

Enhancement of transport

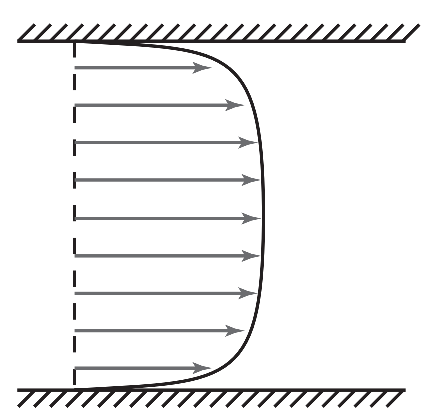
$$\frac{DU^\alpha}{Dt} \equiv \left(\frac{\partial}{\partial t} + U^a \frac{\partial}{\partial x^a}\right) U^\alpha = -\frac{\partial P}{\partial x^\alpha} - \frac{\partial}{\partial x^a} \langle u'^a u'^\alpha \rangle + \nu \frac{\partial^2 U^\alpha}{\partial x^{a2}}$$

Reynolds stress $$\langle u'_\alpha u'_\beta \rangle = \frac{2}{3} K \delta_{\alpha\beta} - \nu_{\mathrm{T}} \left(\frac{\partial U_\alpha}{\partial x_\beta} + \frac{\partial U_\beta}{\partial x_\alpha} \right)$$ (Model)

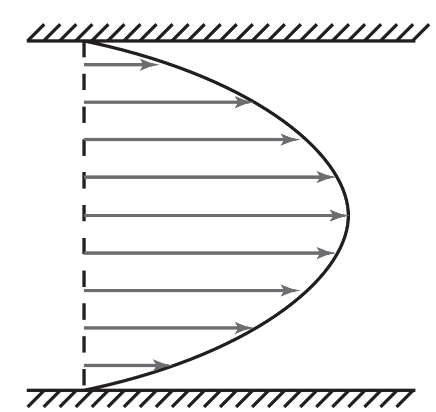
ν_{T}: eddy viscosity (turbulent viscosity) (Boussinesq, 1877)

$$\rightarrow \quad \frac{\partial U_\alpha}{\partial t} + U_a \frac{\partial U_\alpha}{\partial x_a} = -\frac{\partial P}{\partial x_\alpha} + \frac{\partial}{\partial x_a} \left[(\nu + \nu_{\mathrm{T}}) \left(\frac{\partial U_\alpha}{\partial x_a} + \frac{\partial U_a}{\partial x_\alpha} \right) \right]$$

- enhancing transport
- spatial and temporal dependence



Turbulent



Laminar

Transport coefficients

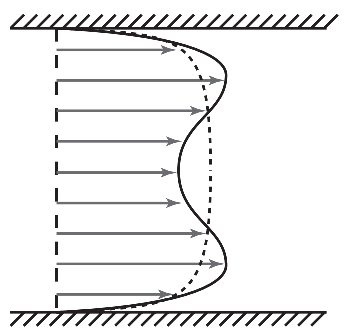
An example: Turbulent eddy viscosity ν_{T}

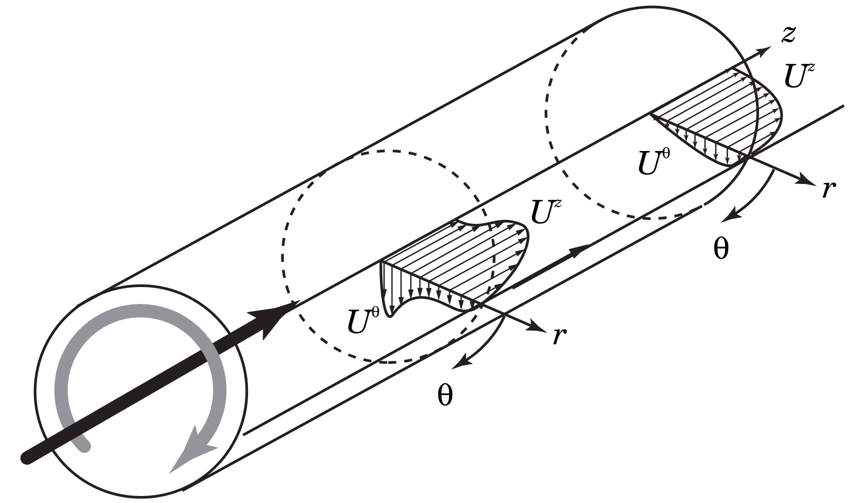
- Parameters $\nu_{\mathrm{T}} = \nu_{\mathrm{T0}}$
- Mixing length $\nu_{\mathrm{T}} = u\ell$
- Turbulence energy $\nu_{\mathrm{T}} = \tau u^2 = \tau K \qquad K = \left\langle \mathbf{u}'^2 \right\rangle /2$
- Propagators $\nu_{\mathrm{T}} = \dfrac{7}{15} \int \mathrm{d}\mathbf{k} \int_{-\infty}^{t} d\tau_1 \; G(k;\tau,\tau_1) Q(k;\tau,\tau_1)$
- Transport equations $\nu_{\mathrm{T}} = K\tau = C_\mu K \dfrac{K}{\varepsilon}$

$$\left(\frac{\partial}{\partial t} + U_a \frac{\partial}{\partial x_a} \right) K = P_K - \varepsilon + \nabla \cdot \left(\frac{\nu_{\mathrm{T}}}{\sigma_K} \nabla K \right)$$
$$\left(\frac{\partial}{\partial t} + U_a \frac{\partial}{\partial x_a} \right) \varepsilon = C_{\varepsilon 1} \frac{\varepsilon}{K} P_K - C_{\varepsilon 2} \frac{\varepsilon}{K} \varepsilon + \nabla \cdot \left(\frac{\nu_{\mathrm{T}}}{\sigma_\varepsilon} \nabla \varepsilon \right)$$

Suppression of transport

Turbulent swirling pipe flow





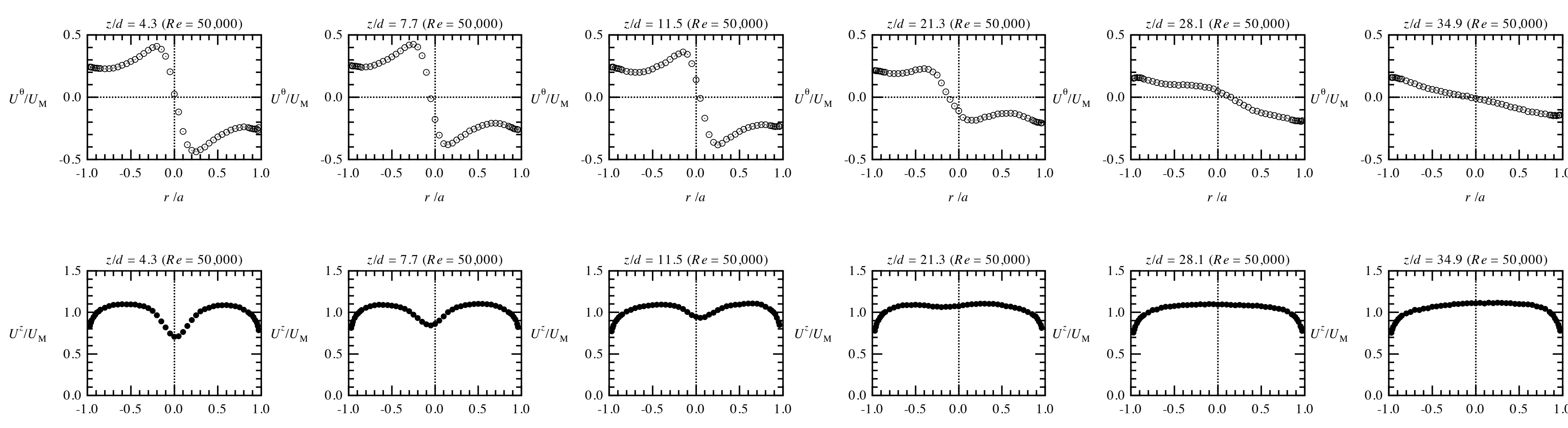
Large-scale structure again! ⟵ Additional symmetry breakage

Experimental studies (Kitoh, 1991; Steenbergen, 1995)

upstream ⟶ downstream

circumferential velocity

axial velocity



II. Turbulence modelling based on statistical theory

Vortex generation

Vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \underbrace{\frac{\nabla \rho \times \nabla p}{\rho^2}}_{\text{baroclinicity}} + \nu \nabla^2 \boldsymbol{\omega}$$

cf., Biermann battery $-\dfrac{\nabla n_e \times \nabla p_e}{n_e^2 e}$

Mean vorticity

$$\boldsymbol{\Omega} = \nabla \times \mathbf{U}$$

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \nabla \times (\mathbf{U} \times \boldsymbol{\Omega}) + \nabla \times \underbrace{\langle \mathbf{u}' \times \boldsymbol{\omega}' \rangle}_{\mathbf{V}_{\mathrm{M}}} + \nu \nabla^2 \boldsymbol{\Omega}$$

vortexmotive force

cf., Mean magnetic field

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times \underline{\langle \mathbf{u}' \times \mathbf{b}' \rangle} + \eta \nabla^2 \mathbf{B}$$

electromotive force

Reynolds stress $\mathcal{R}^{ij} = \langle u'^i u'^j \rangle$

$$V_{\mathrm{M}}^i = -\frac{\partial \mathcal{R}^{ij}}{\partial x^j} + \frac{\partial K}{\partial x^i}$$

Modelling in dynamos

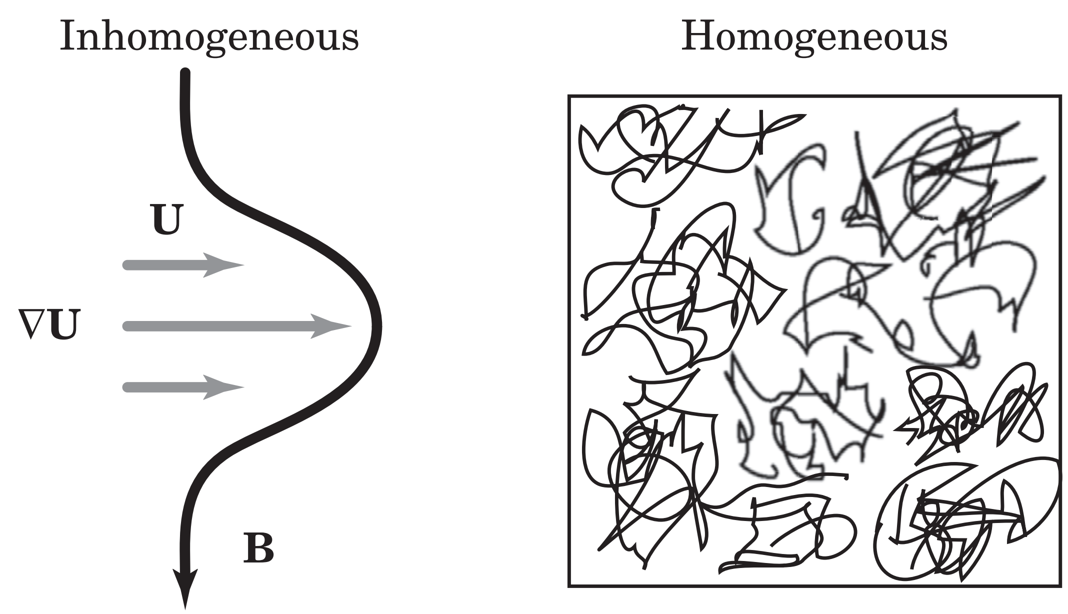
$$\langle \mathbf{u}' \times \mathbf{b}' \rangle^{\alpha} = \alpha^{\alpha a} B^{a} + \beta^{\alpha a b} \frac{\partial B^{a}}{\partial x^{b}} + \cdots$$

Mean field

$$\mathbf{b} = \mathbf{B} + \mathbf{b}', \quad \mathbf{B} = \langle \mathbf{b} \rangle$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times \langle \mathbf{u}' \times \mathbf{b}' \rangle + \eta \nabla^{2} \mathbf{B}$$

$(\mathbf{B} \cdot \nabla)\mathbf{U}$ → differential rotation, “Ω effect”



Turbulence

$$\mathbf{U} = \mathbf{U}_0(\text{constant}) \ \text{ or } \ \mathbf{0}$$

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{u}' = (\mathbf{B} \cdot \nabla)\mathbf{b}' + (\mathbf{b}' \cdot \nabla)\mathbf{B} \cancel{- (\mathbf{u}' \cdot \nabla)\mathbf{U}} + \cdots$$

$$\frac{\partial \mathbf{b}'}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{b}' = (\mathbf{B} \cdot \nabla)\mathbf{u}' - (\mathbf{u}' \cdot \nabla)\mathbf{B} \cancel{+ (\mathbf{b}' \cdot \nabla)\mathbf{U}} + \cdots$$

→ $$\langle \mathbf{u}' \times \mathbf{b}' \rangle^{\alpha} = \alpha^{\alpha a} B^{a} + \beta^{\alpha a b} \frac{\partial B^{a}}{\partial x^{b}} + \cdots$$ “Ansatz”

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{u}' = (\mathbf{B}\cdot\nabla)\mathbf{b}' + (\mathbf{b}'\cdot\nabla)\mathbf{B} - (\mathbf{u}'\cdot\nabla)\mathbf{U} + \cdots$$

$$\frac{\partial \mathbf{b}'}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{b}' = (\mathbf{B}\cdot\nabla)\mathbf{u}' - (\mathbf{u}'\cdot\nabla)\mathbf{B} + (\mathbf{b}'\cdot\nabla)\mathbf{U} + \cdots$$

$$\left\langle \frac{\partial \mathbf{u}'}{\partial t} \times \mathbf{b}' \right\rangle + \left\langle \mathbf{u}' \times \frac{\partial \mathbf{b}'}{\partial t} \right\rangle = \cdots$$

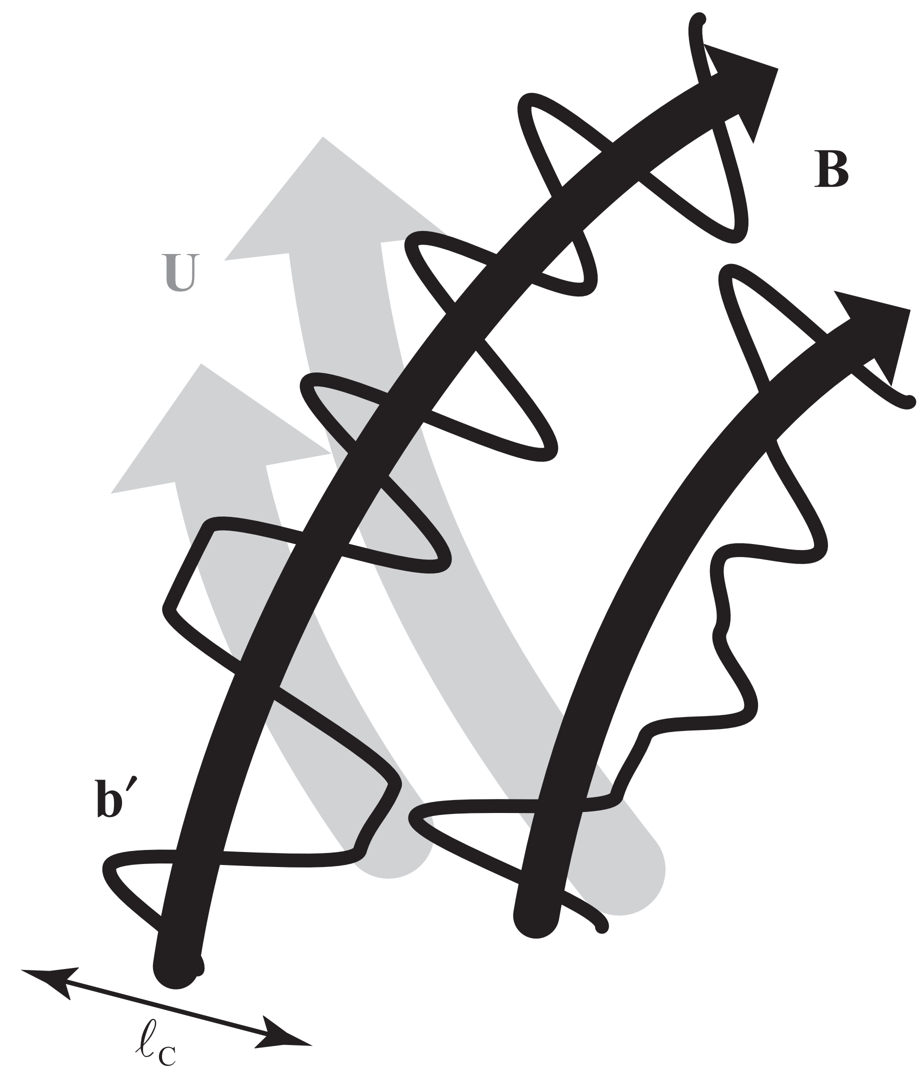
$$\tau\langle \mathbf{u}' \times [(\mathbf{b}'\cdot\nabla)\mathbf{U}] + [(\mathbf{u}'\cdot\nabla)\mathbf{U}] \times \mathbf{b}' \rangle^{\alpha}$$

$$= \epsilon^{\alpha a b}\tau\langle u'^{a} b'^{c}\rangle \frac{\partial U^b}{\partial x^c} - \epsilon^{\alpha b a}\tau\langle b'^{a} u'^{c}\rangle \frac{\partial U^b}{\partial x^c}$$

$$= \tau\left(\langle u'^{a} b'^{c}\rangle + \langle u'^{c} b'^{a}\rangle\right)\epsilon^{\alpha a b}\frac{\partial U^b}{\partial x^c}$$

$$\longrightarrow \quad \langle \mathbf{u}' \times \mathbf{b}' \rangle = \cdots + \tau\underline{\langle \mathbf{u}'\cdot\mathbf{b}'\rangle}\nabla\times\mathbf{U} + \cdots$$

cross helicity



$$R^{ij} = R^{ij}_{\rm non-diff} + N_{ijk\ell}\frac{\partial U^k}{\partial x_\ell}$$

Theoretical formulation

Yoshizawa, 1984: mirror-symmetric case
Yokoi & Yoshizawa, 1993: non-mirror-symmetric case

DIA — A closure theory (propagator renormalization) for homogeneous isotropic turbulence

Multiple-scale analysis — Fast and slowly varying fields

- Introduction of two scales
- Fourier transform of the fast variables
- Scale-parameter expansion
- Introduction of the Green's function
- Statistical assumptions on the basic fields
- Calculation of the statistical quantities using the DIA

(i) Introduction of two scales

Fast and slow variables

$$\boldsymbol{\xi} = \mathbf{x},\ \mathbf{X} = \delta\mathbf{x};\ \tau = t,\ T = \delta t$$

Slow variables ***X*** and *T* change only when ***x*** and *t* change much.

$$f = F\left(\mathbf{X}; T\right) + f'\left(\boldsymbol{\xi}, \mathbf{X}; \tau, T\right)$$

$$\nabla = \nabla_{\boldsymbol{\xi}} + \delta\nabla_{\mathbf{X}};\quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \delta\frac{\partial}{\partial T}$$

Velocity-fluctuation equation

$$\frac{\partial u'_\alpha}{\partial \tau} + U_a\frac{\partial u'_\alpha}{\partial \xi_a} + \frac{\partial}{\partial \xi_a}u'_a u'_\alpha + \frac{\partial p'}{\partial \xi_\alpha} - \nu\nabla_\xi^2 u'_\alpha$$
$$= \delta\left(-u'_a\frac{\partial U_\alpha}{\partial X_a} - \frac{Du'_\alpha}{DT} - \frac{\partial p'}{\partial X_\alpha} - \frac{\partial}{\partial X_a}\left(u'_a u'_\alpha - R_{a\alpha} + 2\nu\frac{\partial^2 u'_\alpha}{\partial X_a \partial \xi_a}\right)\right)$$
$$+\delta^2\left(\nu\nabla_X^2 u'_\alpha\right)$$

$$\frac{\partial u'_a}{\partial \xi_a} + \delta\frac{\partial u'_a}{\partial X_a} = 0$$ where $$\frac{D}{DT} = \frac{\partial}{\partial T} + \mathbf{U}\cdot\nabla_X$$

(ii) Fourier transform of the fast variables

The fluctuation fields are homogeneous with respect to the fast variables:

$$f'(\boldsymbol{\xi}, \mathbf{X}; \tau, T) = \int d\mathbf{k} f'(\mathbf{k}, \mathbf{X}; \tau, T) \exp(-i\mathbf{k} \cdot (\boldsymbol{\xi} - \mathbf{U}\tau))$$

(iii) Scale-parameter expansion

$$f' = f'_0 + \delta f'_1 + \delta^2 f'_2 + \cdots = \sum_n \delta^n f'_n$$

Eliminating the pressure term, we have

$$\frac{\partial u'_{0\alpha}(\mathbf{k};\tau)}{\partial \tau} + \nu k^2 u'_{0\alpha}(\mathbf{k};\tau)$$
$$-iM_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_{0a}(\mathbf{p};\tau) u'_{0b}(\mathbf{q};\tau) = 0$$

(iv) Introduction of the Green's function

$$\frac{\partial G'_{\alpha\beta}(\mathbf{k};\tau,\tau')}{\partial \tau} + \nu k^2 G'_{\alpha\beta}(\mathbf{k};\tau,\tau')$$
$$-2iM^{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_{0a}(\mathbf{p};\tau) G'_{b\beta}(\mathbf{q};\tau,\tau')$$
$$= D_{\alpha\beta}(\mathbf{k}) \delta(\tau - \tau')$$

1st-order field

$$\frac{\partial u'_{1\alpha}(\mathbf{k};\tau)}{\partial \tau} + \nu k^2 u'_{1\alpha}(\mathbf{k};\tau)$$

$$-2iM_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_{0a}(\mathbf{p};\tau) u'_{S1b}(\mathbf{q};\tau)$$

$$= -D_{\alpha b}(\mathbf{k}) u'_{0a}(\mathbf{k};\tau) \frac{\partial U_b}{\partial X_a} - D_{\alpha a}(\mathbf{k}) \frac{Du'_{0a}(\mathbf{k};\tau)}{DT_{\mathrm{I}}}$$

$$+2M_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{q_b}{q^2} u'_{0a}(\mathbf{p};\tau) \frac{\partial u'_{0c}(\mathbf{q};\tau)}{\partial X_{\mathrm{I}c}}$$

$$-D_{\alpha d}(\mathbf{k}) M_{abcd}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{\partial}{\partial X_{\mathrm{I}c}} \left(u'_{0a}(\mathbf{p};\tau) u'_{0b}(\mathbf{q};\tau)\right)$$

$$\mathbf{u}'_1(\mathbf{k};\tau) = \mathbf{u}'_{\mathrm{S}1}(\mathbf{k};\tau) - i\frac{\mathbf{k}}{k^2} \frac{\partial u'_{0a}}{\partial X_{\mathrm{I}a}}$$

$$\mathbf{k} \cdot \mathbf{u}'_{\mathrm{S}1}(\mathbf{k};\tau) = 0 \qquad M_{abcd}(\mathbf{k}) = \frac{1}{2}\delta_{ac}\delta_{bd} + \frac{1}{2}\delta_{ad}\delta_{bc} - \frac{k_a k_b}{k^2}\delta_{cd}$$

Green's function

$$\frac{\partial G'_{\alpha\beta}(\mathbf{k};\tau,\tau')}{\partial \tau} + \nu k^2 G'_{\alpha\beta}(\mathbf{k};\tau,\tau')$$

$$-2iM^{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_{0a}(\mathbf{p};\tau) G'_{b\beta}(\mathbf{q};\tau,\tau')$$

$$= D_{\alpha\beta}(\mathbf{k}) \delta(\tau-\tau')$$

Formal solution in terms of $G'_{\alpha\beta}(\mathbf{k};\tau,\tau')$

$$
\begin{aligned}
u'_{\mathrm{S}1\alpha}(\mathbf{k};\tau) = & -\frac{\partial U_b}{\partial X_a}\int_{-\infty}^{\tau} d\tau_1 G'_{\alpha b}(\mathbf{k};\tau,\tau_1) u'_{0a}(\mathbf{k};\tau_1) \\
& - \int_{-\infty}^{\tau} d\tau_1 G'_{\alpha a}(\mathbf{k};\tau,\tau_1)\frac{Du'_{0a}(\mathbf{k};\tau_1)}{DT_{\mathrm{I}}} \\
& + 2M_{dab}(\mathbf{k})\iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}\int_{-\infty}^{\tau} d\tau_1 G'_{\alpha d}(\mathbf{k};\tau,\tau_1) \\
& \quad \times \frac{q_b}{q^2} u'_{0a}(\mathbf{p};\tau_1)\frac{\partial u'_{0c}(\mathbf{q};\tau_1)}{\partial X_{\mathrm{I}c}} \\
& - M_{abcd}(\mathbf{k})\iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}\int_{-\infty}^{\tau} d\tau_1 G'_{\alpha d}(\mathbf{k};\tau,\tau_1) \\
& \quad \times \frac{\partial}{\partial X_{\mathrm{I}c}}\left(u'_{0a}(\mathbf{p};\tau_1) u'_{0b}(\mathbf{q};\tau_1)\right)
\end{aligned}
$$

(v) Statistical assumptions on the basic field

Basic field: homogeneous isotropic but non-mirror-symmetric

$$\frac{\langle u'_{0\alpha}(\mathbf{k};\tau) u'_{0\beta}(\mathbf{k};\tau) \rangle}{\delta(\mathbf{k}+\mathbf{k}')} = D_{\alpha\beta}(\mathbf{k}) Q_0(k;\tau,\tau') + \frac{i}{2}\frac{k_a}{k^2}\epsilon_{\alpha\beta a} H_0(k;\tau,\tau')$$

(vi) Calculation of the statistical quantities by DIA

$$\left\langle {u'}^\alpha {u'}^\beta \right\rangle = \left\langle {u'_\mathrm{B}}^\alpha {u'_\mathrm{B}}^\beta \right\rangle + \left\langle {u'_\mathrm{B}}^\alpha {u'_{01}}^\beta \right\rangle + \left\langle {u'_{01}}^\alpha {u'_\mathrm{B}}^\beta \right\rangle + \cdots$$
$$+ \left\langle {u'_\mathrm{B}}^\alpha {u'_{10}}^\beta \right\rangle + \left\langle {u'_{10}}^\alpha {u'_\mathrm{B}}^\beta \right\rangle + \cdots$$

$$\left\langle {u'}^\alpha {u'}^\beta \right\rangle_\mathrm{D} = -\nu_\mathrm{T} \mathcal{S}^{\alpha\beta} + \left[\Gamma^\alpha \left(\Omega^\beta + 2\omega_\mathrm{F}^\beta \right) + \Gamma^\beta \left(\Omega^\alpha + 2\omega_\mathrm{F}^\alpha \right) \right]_\mathrm{D}$$

where $$\mathcal{S}^{\alpha\beta} = \frac{\partial U^\alpha}{\partial x^\beta} + \frac{\partial U^\beta}{\partial x^\alpha} - \frac{2}{3}\nabla\cdot\mathbf{U}\delta^{\alpha\beta}$$

mixing length $\nu_\mathrm{T} \sim \tau u^2 \sim u\ell$

Eddy viscosity $$\nu_\mathrm{T} = \frac{7}{15}\int \mathrm{d}\mathbf{k} \int_{-\infty}^{t} d\tau_1 \; G(k;\tau,\tau_1) Q(k;\tau,\tau_1)$$

Helicity-related coefficient $$\mathbf{\Gamma} = \frac{1}{30}\int k^{-2}\mathrm{d}\mathbf{k} \int_{-\infty}^{t} d\tau_1 \; G(k;\tau,\tau_1) \nabla H(k;\tau,\tau_1)$$

helicity inhomogeneity is essential

Eddy viscosity + Helicity model

Reynolds stress

Yokoi & Yoshizawa (1993) Phys. Fluids A**5**, 464

$$\mathcal{R}_{\alpha\beta} \equiv \langle u'_\alpha u'_\beta \rangle$$

$$= \frac{2}{3} K \delta_{\alpha\beta} - \nu_\mathrm{T} \left(\frac{\partial U_\alpha}{\partial x_\beta} + \frac{\partial U_\beta}{\partial x_\alpha} \right) + \eta \left[\Omega_\alpha \frac{\partial H}{\partial x_\beta} + \Omega_\beta \frac{\partial H}{\partial x_\alpha} - \frac{2}{3} \delta_{\alpha\beta} \left(\mathbf{\Omega} \cdot \nabla \right) H \right]$$

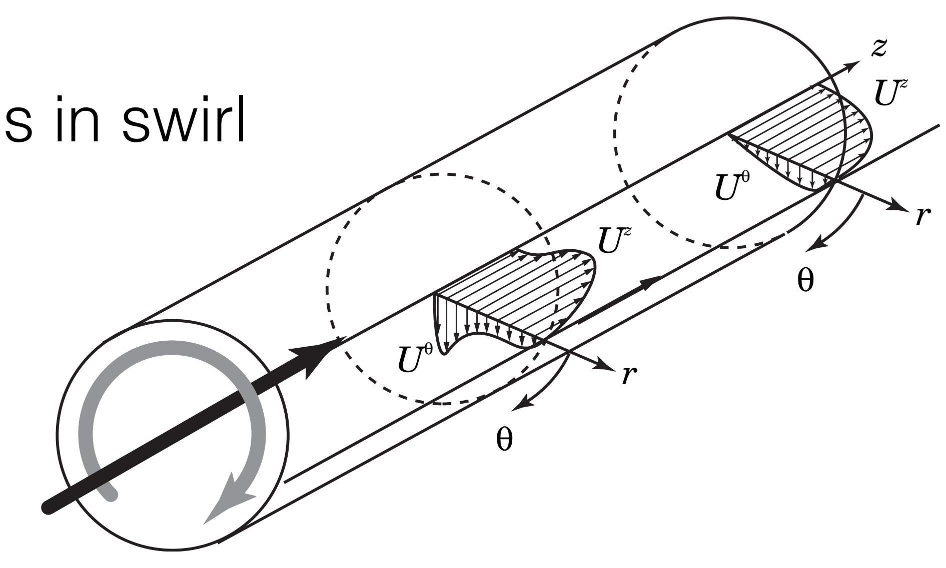
$$\nu_\mathrm{T} = C_\nu \tau K, \quad \tau = K/\epsilon, \quad \eta = C_H \tau (K^3/\epsilon^2)$$

Turbulence quantities

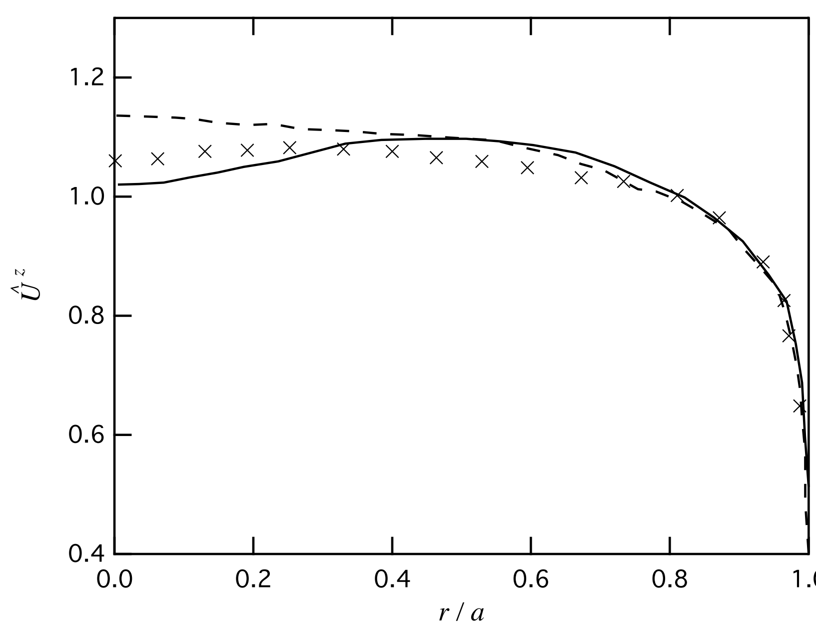
$$K \equiv \frac{1}{2} \langle \mathbf{u}'^2 \rangle, \quad \epsilon \equiv \nu \left\langle \frac{\partial u'_b}{\partial x_a} \frac{\partial u'_b}{\partial x_a} \right\rangle,$$

$$H \equiv \langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle, \quad \epsilon_H \equiv 2\nu \left\langle \frac{\partial u'_b}{\partial x_a} \frac{\partial \omega'_b}{\partial x_a} \right\rangle$$

Velocity profiles in swirl

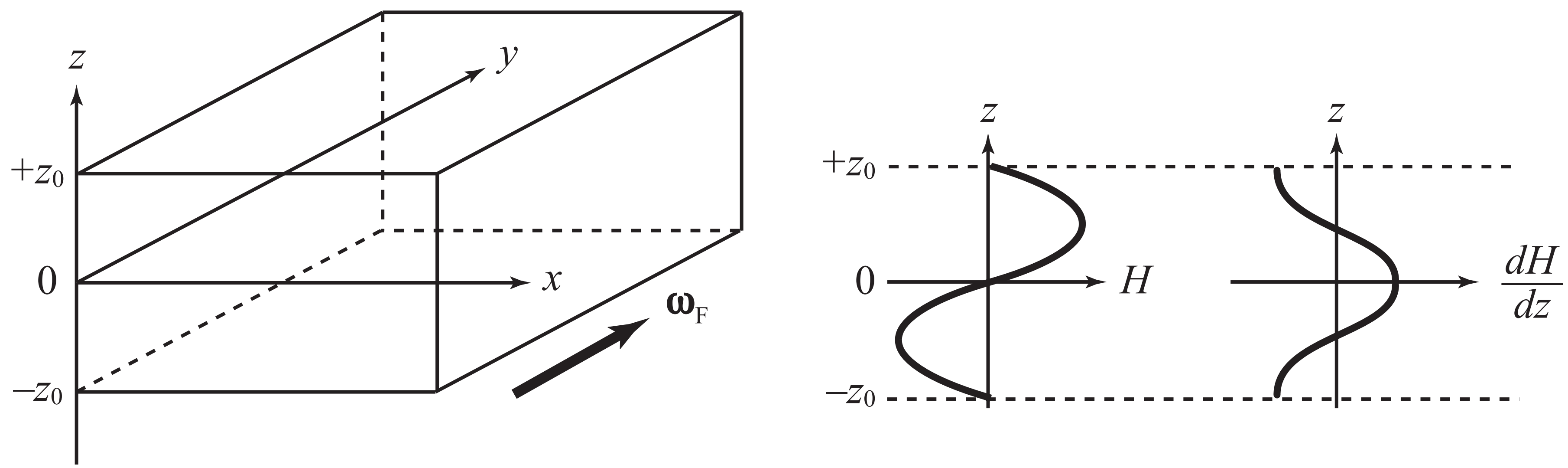


Helicity turbulence model



III. Global flow generation

DNS set-up



Set-up of the turbulence and rotation $\boldsymbol{\omega}_F$ (left), the schematic spatial profile of the turbulent helicity H (= $\langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle$) (center) and its derivative dH/dz (right).

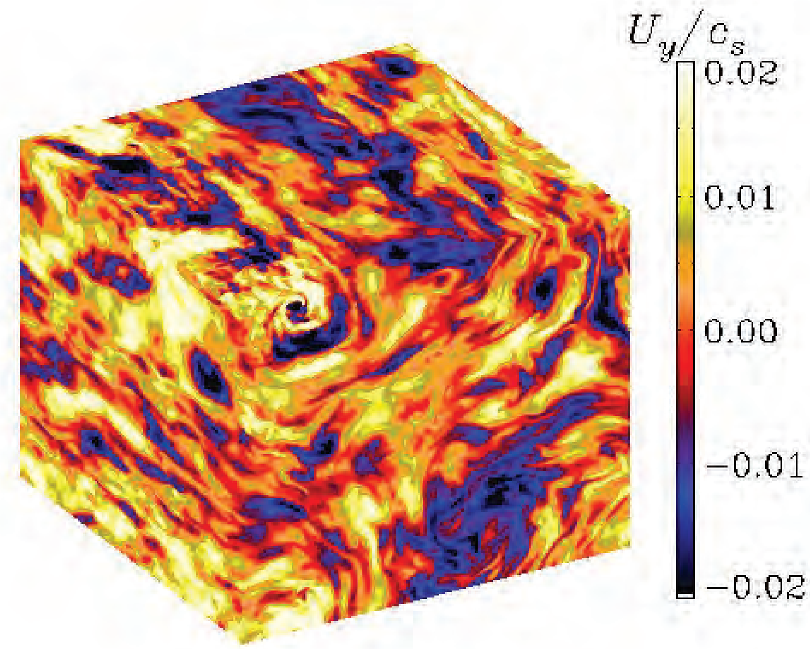
Rotation $\boldsymbol{\omega}_F = (\omega_F^x, \omega_F^y, \omega_F^z) = (0, \omega_F, 0)$

Inhomogeneous turbulent helicity $H(z) = H_0 \sin(\pi z / z_0)$

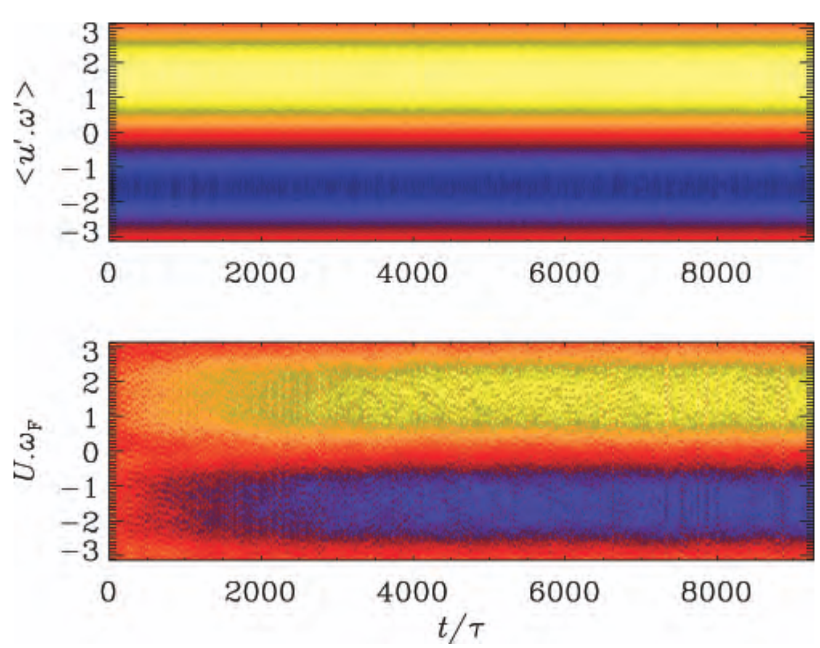
Run	k_f/k_1	Re	Co	$\eta/(\nu_T \tau^2)$
A	15	60	0.74	0.22
B1	5	150	2.6	0.27
B2	5	460	1.7	0.27
B3	5	980	1.6	0.51
C1	30	18	0.63	0.50
C2	30	80	0.55	0.03
C3	30	100	0.46	0.08

Summary of DNS results

Global flow generation



Axial flow component U^y on the periphery of the domain



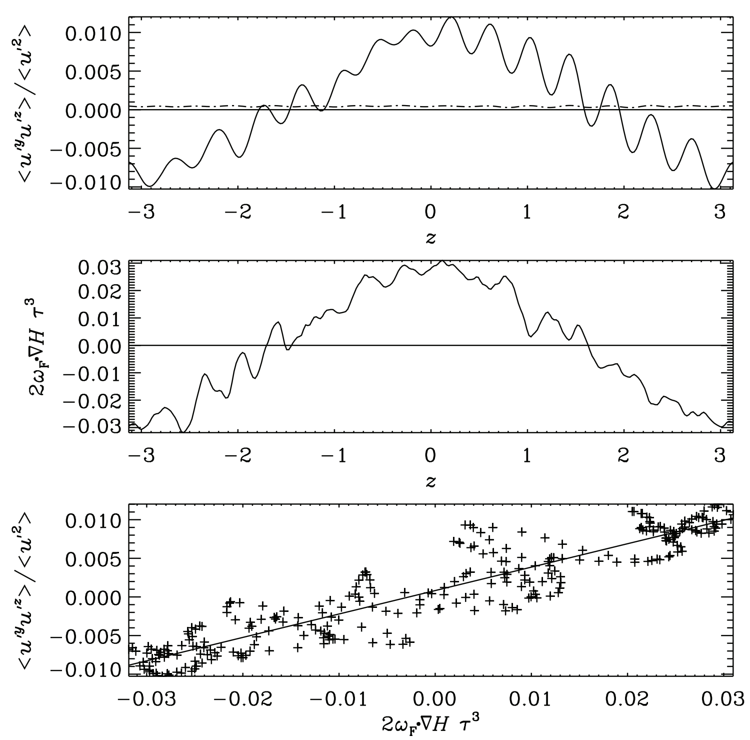
Turbulent helicity $\langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle$ (top) and mean-flow helicity $\mathbf{U} \cdot 2\boldsymbol{\omega}_F$ (bottom)

Reynolds stress

$$\left\langle u'^{\alpha} u'^{\beta} \right\rangle_{\mathrm{D}} = -\nu_{\mathrm{T}} \mathcal{S}^{\alpha\beta} + \left[\Gamma^{\alpha} \left(\Omega^{\beta} + 2\omega_{\mathrm{F}}^{\beta} \right) + \Gamma^{\beta} \left(\Omega^{\alpha} + 2\omega_{\mathrm{F}}^{\alpha} \right) \right]_{\mathrm{D}}$$

Early stage

$$\langle u'^{y} u'^{z} \rangle = \eta 2\omega_{\mathrm{F}}^{y} \frac{\partial H}{\partial z}$$

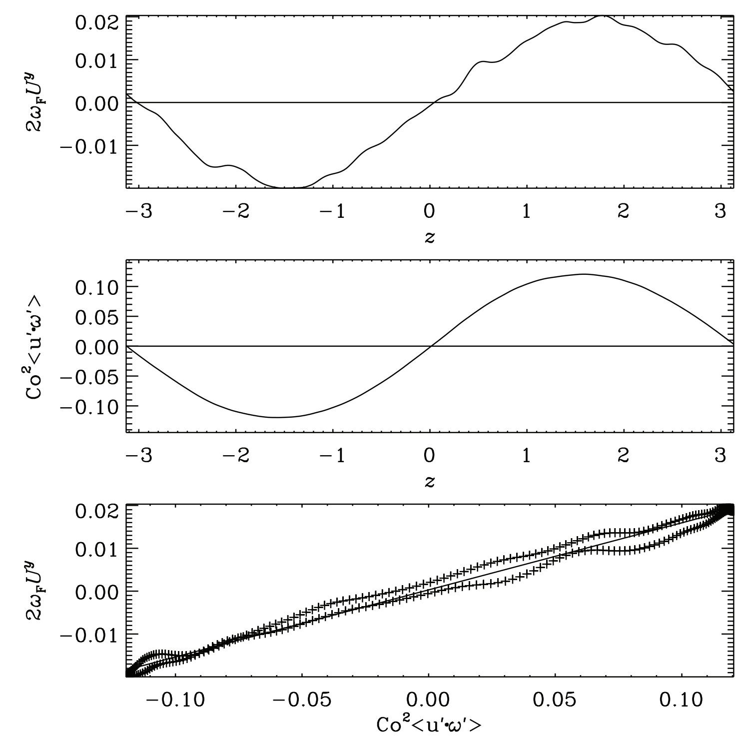


Reynolds stress $\langle u'^{y} u'^{z} \rangle$ (top), helicity-effect term $(\nabla H)^{z}\, 2\omega_{\mathrm{F}}^{y}$ (middle), and their correlation (bottom).

Developed stage

$$\langle u'^{y} u'^{z} \rangle = -\nu_{\mathrm{T}} \frac{\partial U^{y}}{\partial z} + \eta 2\omega_{\mathrm{F}}^{y} \frac{\partial H}{\partial z}$$

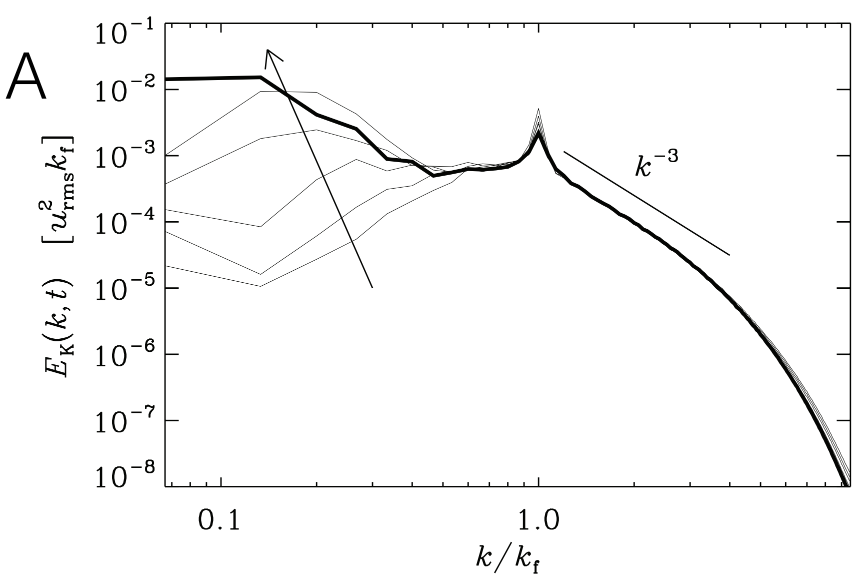
$$U^{y} = (\eta / \nu_{\mathrm{T}})\, 2\omega_{\mathrm{F}}^{y} H$$



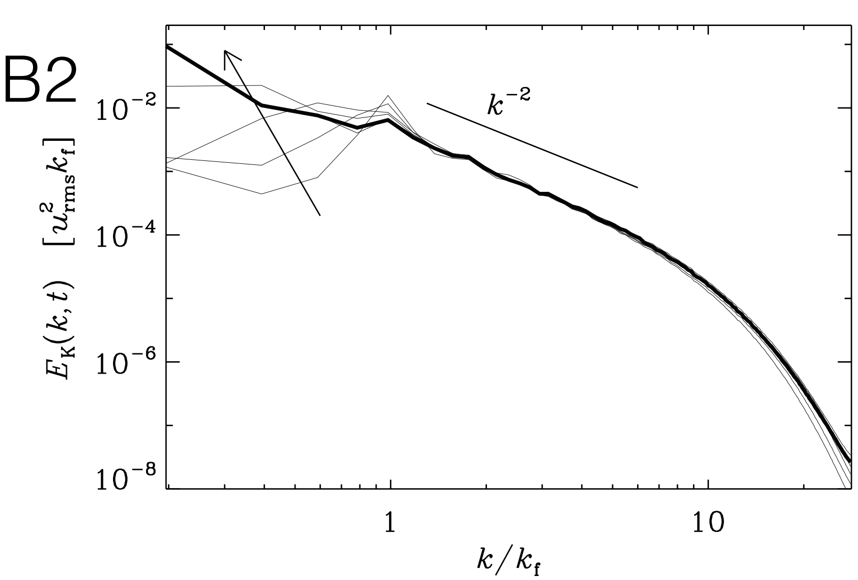
Mean axial velocity U^{y} (top), turbulent helicity multiplied by rotation $2\omega_{\mathrm{F}} H$ (middle), and their correlation (bottom).

Spectra

Run A



Run B2



Physical origin

Reynolds stress $\mathcal{R}^{ij} \equiv \langle u'^i u'^j \rangle$

Vortexmotive force $\mathbf{V}_\mathrm{M} \equiv \langle \mathbf{u}' \times \boldsymbol{\omega}' \rangle$

$$V_\mathrm{M}^i = -\frac{\partial \mathcal{R}^{ij}}{\partial x^j} + \frac{\partial K}{\partial x^i}$$

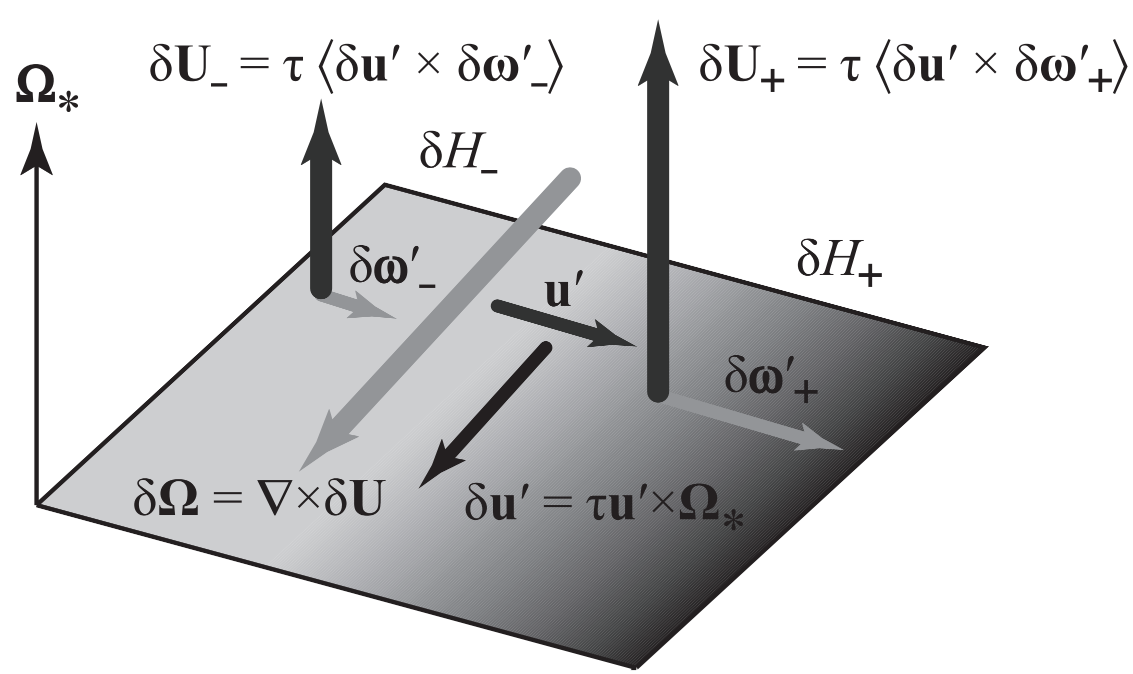
$$\frac{\partial \mathbf{\Omega}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{\Omega}) + \nabla \times \mathbf{V}_\mathrm{M} + \nu \nabla^2 \mathbf{\Omega}$$

$$\mathbf{V}_\mathrm{M} = -D_\Gamma 2\boldsymbol{\omega}_\mathrm{F} - \nu_\mathrm{T} \nabla \times \mathbf{\Omega}$$

$$D_\Gamma = \nabla \cdot \mathbf{\Gamma} \propto \nabla^2 H$$

→ $$\delta \mathbf{U} \sim -(\nabla^2 H)\mathbf{\Omega}_*$$

$$\nabla^2 H \simeq -\frac{\delta H}{\ell^2} = -\frac{\langle \mathbf{u}' \cdot \delta\boldsymbol{\omega}' \rangle}{\ell^2}$$



Reynolds stress evolution

(Inagaki, Yokoi & Hamba, submitted to Phys. Rev. Fluids)

Local helical forcing

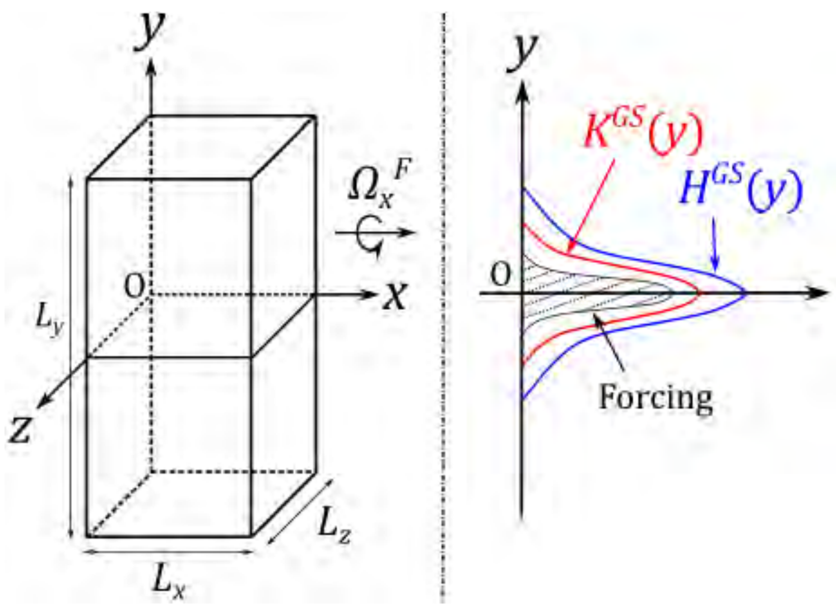
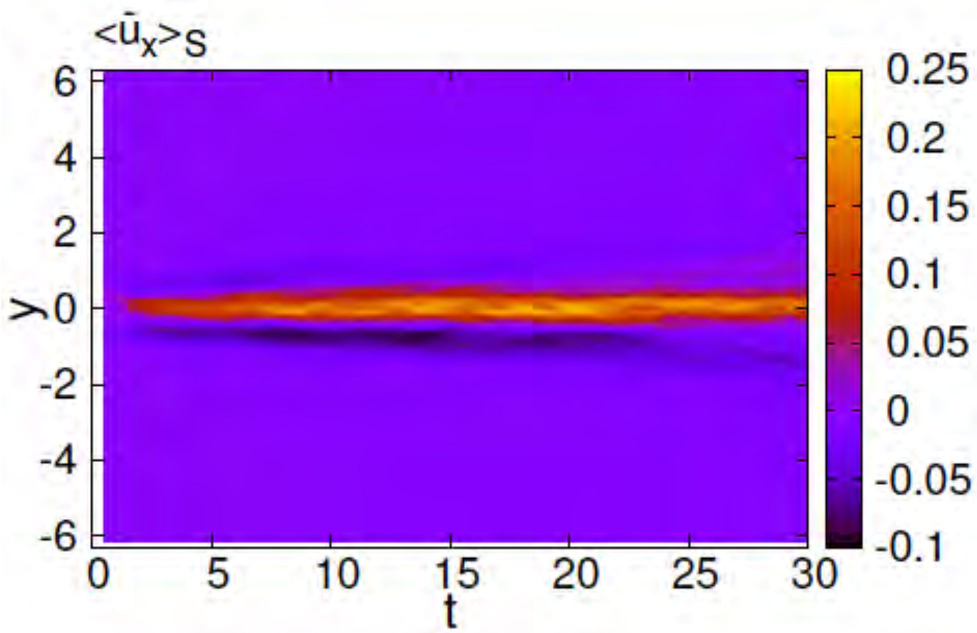
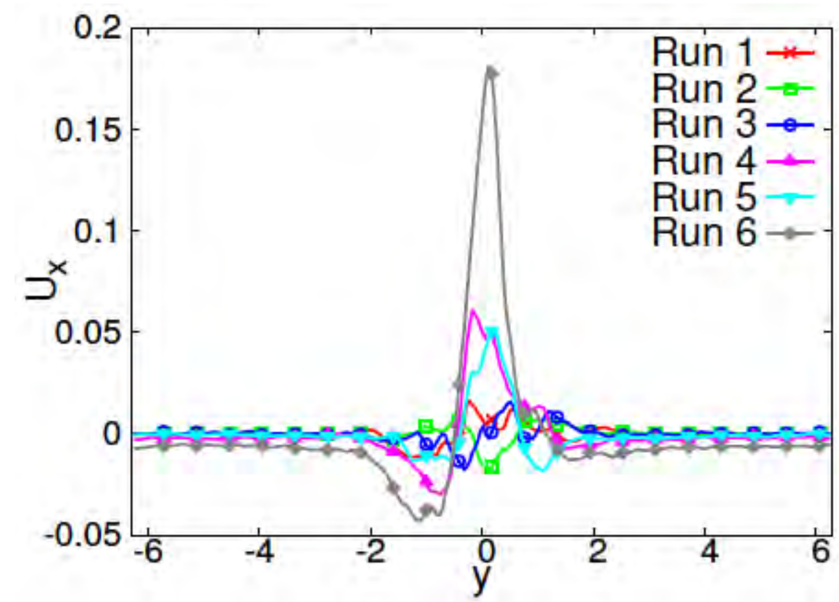


TABLE I. Calculation parameters.

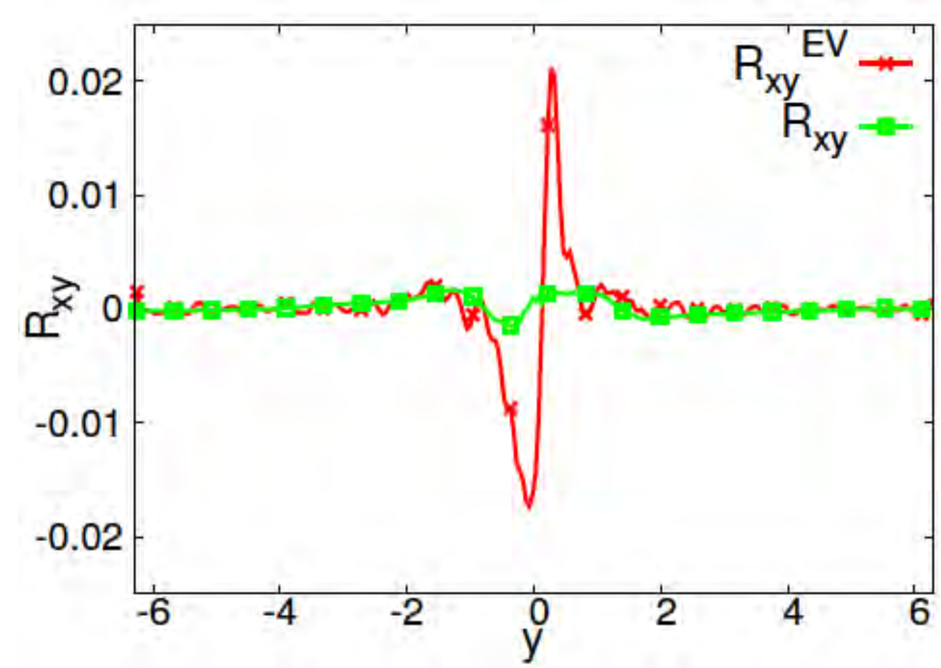
Run	α	Ω_x^F	L_0^{GS}	Ro_0^{GS}
1	0	0	0.506	∞
2	0.5	0	0.547	∞
3	0	5	0.542	0.185
4	0.2	5	0.550	0.182
5	0.5	2	0.544	0.459
6	0.5	5	0.602	0.166





Reynolds-stress budget

$$R_{xy} = \nu_{\mathrm{T}} \frac{\partial U_x}{\partial y} + N_{xy}$$



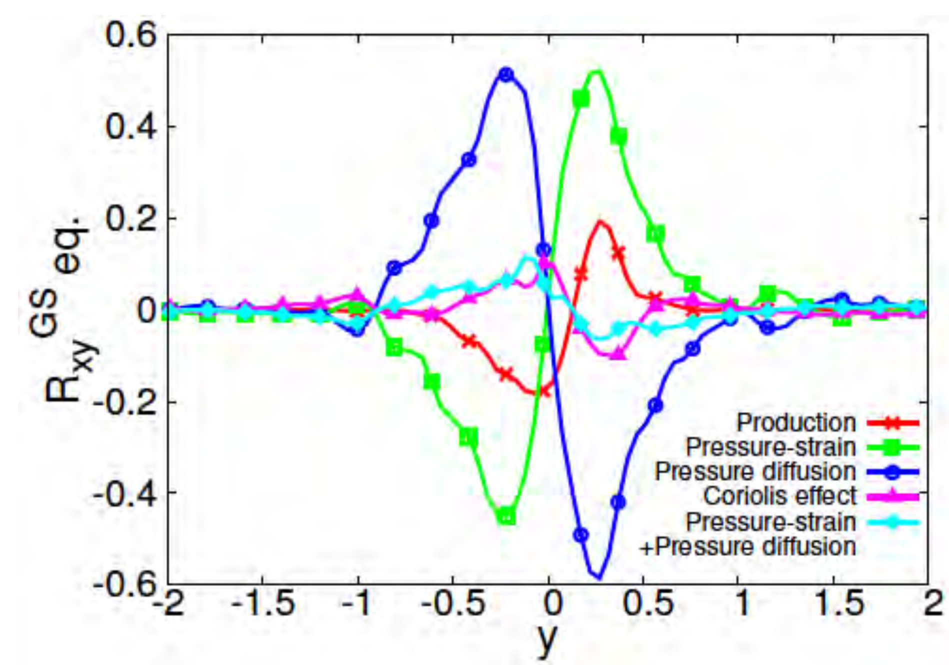
$$\frac{\partial R^{GS}_{xy}}{\partial t} \simeq P^{GS}_{xy} + \Phi^{GS}_{xy} + \Pi^{GS}_{xy} + C^{GS}_{xy} \simeq 0$$

Production $$P^{GS}_{xy} = -\frac{2}{3} K^{GS} \frac{\partial U_x}{\partial y} - B^{GS}_{yy} \frac{\partial U_x}{\partial y} - B^{GS}_{xz} \frac{\partial U_z}{\partial y}$$

Press. strain $$\Phi^{GS}_{xy} = 2 \left\langle \overline{p}' \overline{s}'_{xy} \right\rangle$$

Press. diff. $$\Pi^{GS}_{xy} = -\frac{\partial}{\partial y} \left\langle \overline{p}' \overline{u}'_x \right\rangle$$

Coriolis $$C^{GS}_{xy} = 2 R^{GS}_{xz} \Omega^F_x$$



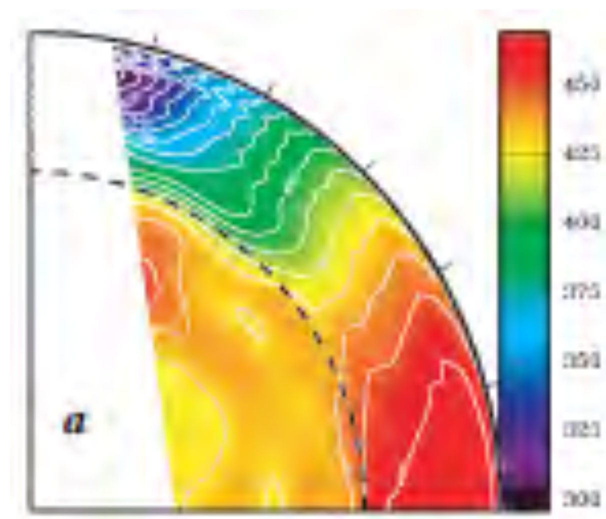
IV. Stellar convection zone

Angular-momentum transport in the solar convection zone

Angular momentum around the rotation axis

$$L = \Gamma r^2 \omega_{\mathrm{F}} + \Gamma r U^\phi \qquad \Gamma = \sin\theta$$

$$\frac{\partial}{\partial t}\rho L + \nabla \cdot (\rho \mathbf{F}_L) = 0$$



Miesch (2005) Liv. Rev. Sol. Phys. 2005-1

Vector flux of angular momentum $\mathbf{F}_L$

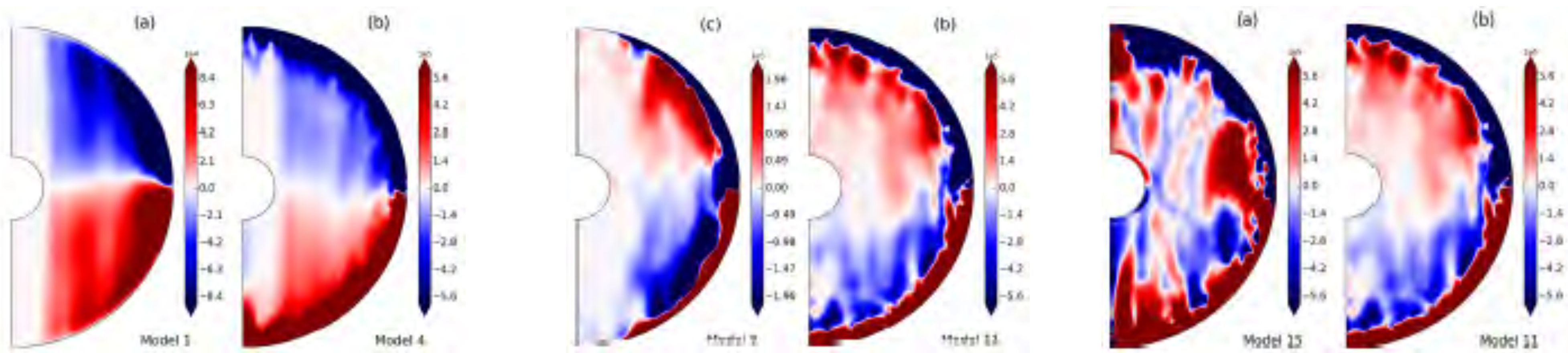
$$F_L^r = L U^r + r\Gamma \mathcal{R}^{r\phi}$$

$$F_L^\theta = L U^\theta + r\Gamma \mathcal{R}^{\theta\phi}$$

Helicity effect $$\mathcal{R}_H^{r\phi} = +\frac{\partial H}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r} r U^\theta - \frac{1}{r}\frac{\partial U^r}{\partial \theta}\right)$$

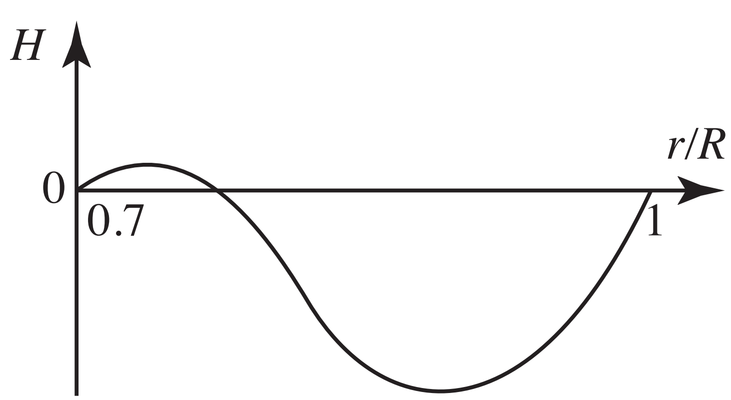
$$\mathcal{R}_H^{\theta\phi} = +\frac{1}{r}\frac{\partial H}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r} r U^\theta - \frac{1}{r}\frac{\partial U^r}{\partial \theta}\right)$$

Helicity effect in the stellar convection zone



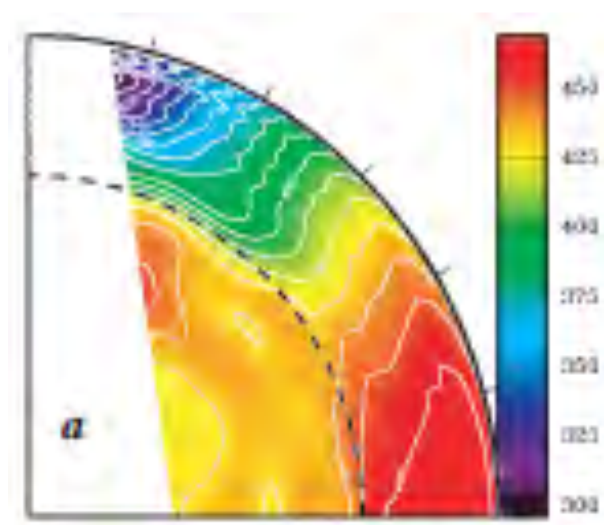
Duarte, et al, (2016) MNRAS **456**, 1708

Schematic helicity distribution



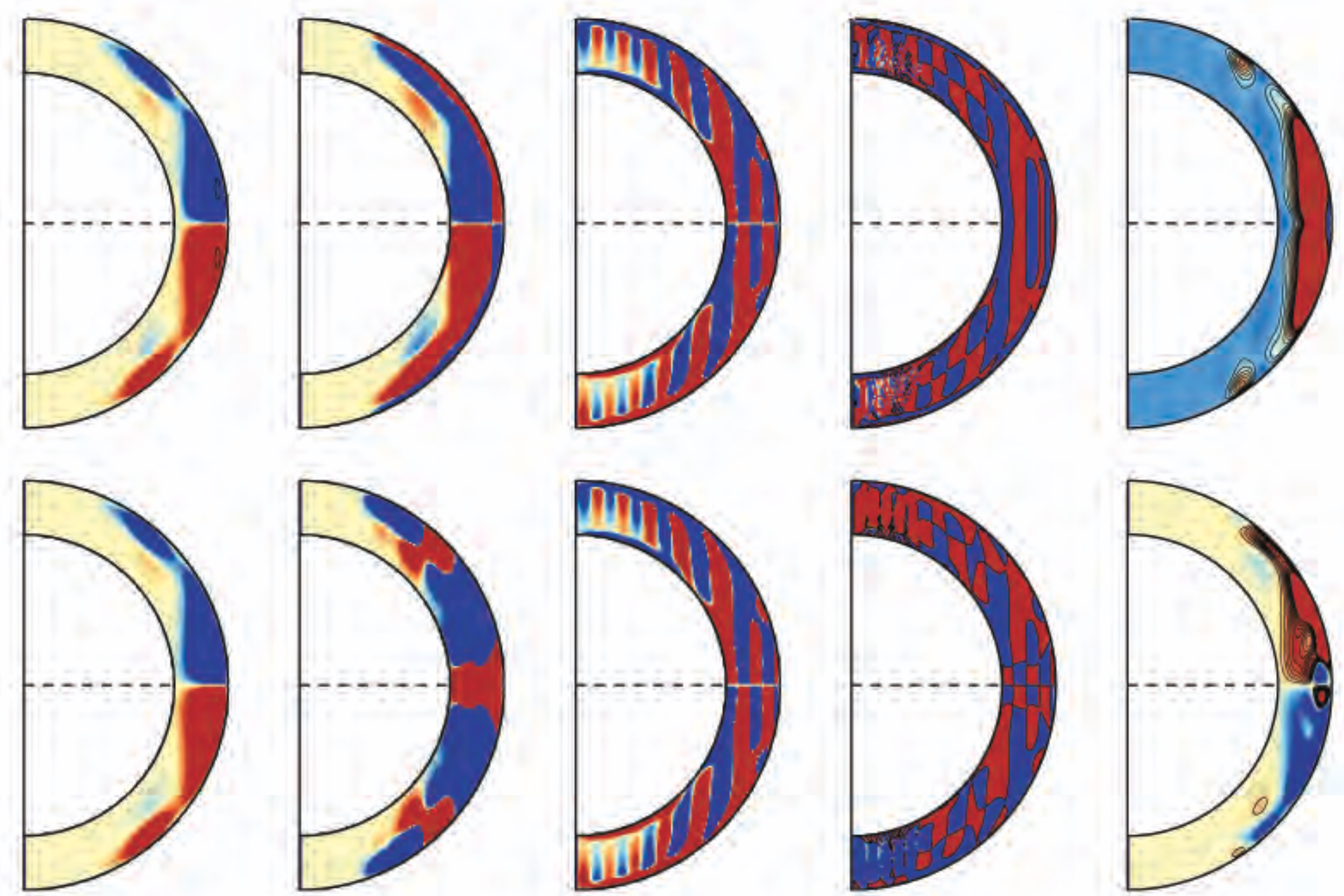
$$\nabla^2 H > 0$$

$$\delta \mathbf{U} \sim -(\nabla^2 H)\mathbf{\Omega}_*$$



Helicity effect in the Reynolds stress

Helicity	Helicity Gradient	Azimuthal Vorticity	Helicity effect	Reynolds stress
$\mathbf{u}\cdot\boldsymbol{\omega}$	$\dfrac{\partial H}{\partial r}$	$\overline{\Omega}^{\phi}$	$\dfrac{\partial H}{\partial r}\overline{\Omega}^{\phi}$	$\overline{u'^r u'^\phi}$



$\mathbf{u}\cdot\boldsymbol{\omega}-\overline{\mathbf{u}}\cdot\overline{\boldsymbol{\omega}}$ $(\equiv H)$	$\dfrac{1}{r}\dfrac{\partial H}{\partial \theta}$	$\overline{\Omega}^{\phi}$	$\dfrac{1}{r}\dfrac{\partial H}{\partial \theta}\overline{\Omega}^{\phi}$	$\overline{u'^\theta u'^\phi}$

(provided by Mark Miesch)

$C_\eta \tau \ell^2 |(\nabla^2 H)\Omega_*|$

Solar parameters

$$v \sim 200 \text{ m s}^{-1} = 2\times 10^4 \text{cm s}^{-1}$$
$$\ell \sim 200 \text{ Mm} = 2\times 10^{10}\text{cm}$$
$$\tau \sim \ell/v \sim 10^6 \text{ s}$$

$r\phi$ component

$$\left|\overline{u'^r u'^\phi}\right| \sim 1.2\times 10^9$$
$$\left|\frac{\partial H}{\partial r}\overline{\Omega}^{\phi}\right| \sim 9.4\times 10^{-15}$$
$$\tau\ell^2\left|\frac{\partial H}{\partial r}\overline{\Omega}^{\phi}\right| \sim 10^{12} \longrightarrow 10^9$$

with $C_\eta = O(10^{-3})$

$\theta\phi$ component

$$\left|\overline{u'^\theta u'^\phi}\right| \sim 5.6\times 10^8$$
$$\left|\frac{1}{r}\frac{\partial H}{\partial \theta}\overline{\Omega}^{\phi}\right| \sim 2.6\times 10^{-15}$$
$$\tau\ell^2\left|\frac{1}{r}\frac{\partial H}{\partial \theta}\overline{\Omega}^{\phi}\right| \sim 10^{11} \longrightarrow 10^8$$

Magnitude same as the Reynolds stress

V. Summary

Summary

In turbulent momentum transport in hydrodynamics

$$\left\langle {u'}^{\alpha} {u'}^{\beta} \right\rangle_{\mathrm{D}} = -\nu_{\mathrm{T}} \mathcal{S}^{\alpha\beta} + \left[\Gamma^{\alpha} \left(\Omega^{\beta} + 2\omega_{\mathrm{F}}^{\beta} \right) + \Gamma^{\beta} \left(\Omega^{\alpha} + 2\omega_{\mathrm{F}}^{\alpha} \right) \right]_{\mathrm{D}}$$

Mean velocity strain (symmetric part of velocity shear)
+ Energy

⟶ Transport enhancement (structure destruction)

Mean absolute vorticity (antisymmetric part of velocity shear)
+ (Inhomogeneous) Helicity

⟶ Transport suppression (structure formation)

N. Yokoi & A. Brandenburg, Phys. Rev. E **34**, 033125 (2016)

N. Yokoi, Geophys. Astrophys. Fluid Dyn. **107**, 114 (2013)

N. Yokoi & A. Yoshizawa, Phys. Fluids A **5**, 464 (1993)

Magnetohydrodynamic Case

$\boldsymbol{\mathcal{S}} = \{\mathcal{S}^{\alpha\beta}\}$: Mean velocity strain

$\boldsymbol{\mathcal{M}} = \{\mathcal{M}^{\alpha\beta}\}$: Mean magnetic strain

Only transport enhancement or structure destruction

$$\mathcal{R}^{\alpha\beta} := -\nu_{\mathrm{K}}\mathcal{S}^{\alpha\beta},$$
$$\mathbf{E}_{\mathrm{M}} := -\beta\mathbf{J}$$

α dynamo

$$\mathcal{R}^{\alpha\beta} := -\nu_{\mathrm{K}}\mathcal{S}^{\alpha\beta} + [\mathbf{\Gamma\Omega}]^{\alpha\beta},$$
$$\mathbf{E}_{\mathrm{M}} := -\beta\mathbf{J} + \alpha\mathbf{B}$$

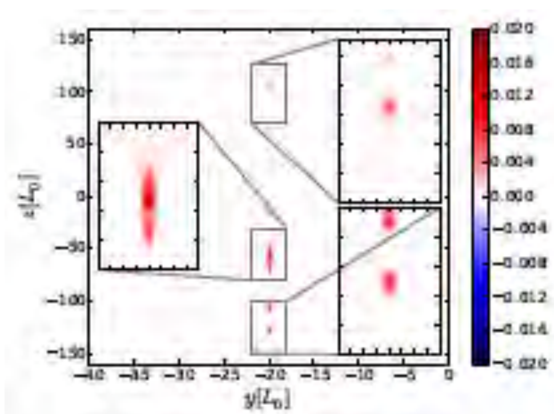
Cross-helicity dynamo

$$\mathcal{R}^{\alpha\beta} := -\nu_{\mathrm{K}}\mathcal{S}^{\alpha\beta} + \nu_{\mathrm{M}}\mathcal{M}^{\alpha\beta},$$
$$\mathbf{E}_{\mathrm{M}} := -\beta\mathbf{J} + \gamma\mathbf{\Omega}$$

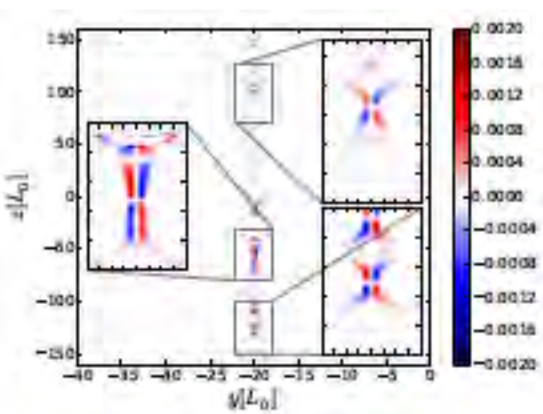
α and cross-helicity dynamo

$$\mathcal{R}^{\alpha\beta} := -\nu_{\mathrm{K}}\mathcal{S}^{\alpha\beta} + \nu_{\mathrm{M}}\mathcal{M}^{\alpha\beta} + [\mathbf{\Gamma\Omega}]^{\alpha\beta},$$
$$\mathbf{E}_{\mathrm{M}} := -\beta\mathbf{J} + \gamma\mathbf{\Omega} + \alpha\mathbf{B}$$

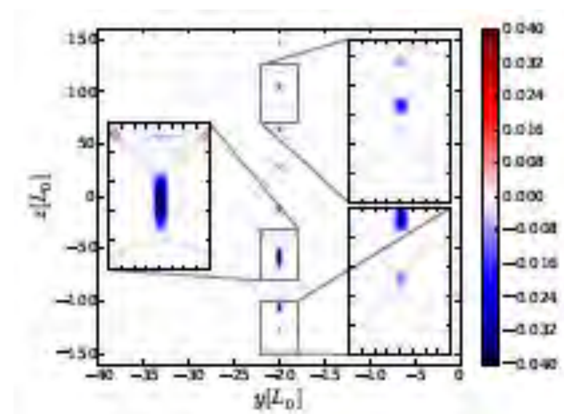
Energy



Cross helicity



Helicity ← Guide field



Widmer, Büchner & Yokoi Phys. Plasmas, **23**, 092304 (2016)

Helicity Thinkshop 3

19-24 November 2017, Tokyo, Japan

Institute of Industrial Science (IIS), Univ. of Tokyo
and
National Astronomical Observatory of Japan (NAOJ)

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Axel Brandenburg, Manolis Georgoulis,
Kirill Kuzanyan, Raffaele Marino, Alexei Pevtsov,
Takashi Sakurai, Dmitry Sokoloff,
Nobumitsu Yokoi (Chair), Hongqi Zhang

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[3] Yokoi, N. & Yoshizawa, A. "Subgrid-scale model with structural effects incorporated through the helicity," in *Progress in Turbulence VII*, pp. 115-121 (2017).

[4] Inagaki, H., Yokoi, N., & Hamba, F. "Mechanism of mean flow generation in rotating turbulence through inhomogeneous helicity," submitted to Phys. Rev. Fluids

[5] Yokoi, N. "Cross helicity and related dynamo," Geophys. Atrophys. Fluid Dyn. **107**, 114-184 (2013).