Upwind Scheme for the Hydrodynamical Equations

- Review of Yesterday's Lecture
- Hydrodynamical Equations
  - Characteristics (= phase velocity)
  - Riemann invariants (=wave amplitude)
- Upwind Scheme
  - Roe Average

Yesterdays' key lessons

- When solving wave equations, we need
  - to rewrite them in the conservation form,
  - to evaluate the numerical flux from the upwind side,

    &

  - to set the CFL number smaller than unity.

Lectures of Today and Tomorrow

- Today we consider one dimensional flow and learn how to solve the hydrodynamic flow. The solution is of first order accuracy.
- Tomorrow we learn how to obtain a solution of second order accuracy (Tomisaka) and how to solve three-dimensional flow and to include gravity (Hanawa).

Today, we learn that

- HD equations are a set of nonlinear wave equations.
- HD equations can be written in the conservation form.
- One dimensional flow has three characteristics and Riemann invariants.
- We can obtain the upwind numerical flux by using the Roe’s formula.

HD equations are a set of nonlinear wave equations (1)

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{v}) = 0 \, ,$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \bullet \nabla)\mathbf{v} + \frac{1}{\rho} \nabla P = \mathbf{g} \, ,$$

$$\rho \, T \, \frac{Ds}{Dt} = \Gamma - \Lambda \, .$$

$$\mathbf{g} = 0$$

$$\Gamma = 0$$

$$\Lambda = 0$$

HD equations are a set of nonlinear wave equations (2)

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{v}) = 0 \,,$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \bullet \nabla)\mathbf{v} + \frac{1}{\rho} \nabla P = 0 \,,$$

$$\rho T \frac{Ds}{Dt} = 0 \,.$$

$$\mathbf{v} \rightarrow v$$

$$\nabla \rightarrow \frac{\partial}{\partial x}$$

HD equations are a set of nonlinear wave equations (3)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0\,,$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0,$$

$$T\left(\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial x}\right) = 0\,.$$

$$P = P(\rho, s)$$

$$a^2 \equiv \frac{dP}{d\rho}$$

HD equations are a set of nonlinear wave equations (4)

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0 , \qquad \Downarrow \times \frac{a}{\rho}$$

$$\frac{a}{\rho}\frac{\partial \rho}{\partial t} + v \frac{a}{\rho} \frac{\partial \rho}{\partial x} + a \frac{\partial v}{\partial x} = 0 , \qquad (1)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{a^2}{\rho}\frac{\partial \rho}{\partial x} = 0, \qquad . \qquad (2)$$

$$\Downarrow \quad (1) + (2)$$

$$\frac{a}{\rho}\frac{\partial \rho}{\partial t} + \frac{\partial v}{\partial t} + (v + a)\left(\frac{a}{\rho}\frac{\partial \rho}{\partial x} + \frac{\partial v}{\partial x}\right) = 0$$

HD equations are a set of nonlinear wave equations (5)

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0 , \quad \Downarrow \times \frac{a}{\rho}$$

$$\frac{a}{\rho}\frac{\partial \rho}{\partial t} + v \frac{a}{\rho} \frac{\partial \rho}{\partial x} + a \frac{\partial v}{\partial x} = 0 , \quad (1)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{a^2}{\rho} \frac{\partial \rho}{\partial x} = 0, \quad . \quad (2)$$

$$\Downarrow \quad (1) \pm (2)$$

$$\frac{a}{\rho}\frac{\partial \rho}{\partial t} \pm \frac{\partial v}{\partial t} + (v \pm a)\left(\frac{a}{\rho} \frac{\partial \rho}{\partial x} \pm \frac{\partial v}{\partial x}\right) = 0$$

HD equations are a set of nonlinear wave equations (6)

$$\frac{\partial J_+}{\partial t} + (v+a)\frac{\partial J_+}{\partial x} = 0 \,,$$

$$\frac{\partial J_-}{\partial t} + (v+a)\frac{\partial J_-}{\partial x} = 0 \,,$$

$$J_+ \equiv \int \frac{a}{\rho} d\rho \ + v \ ,$$

$$J_- \equiv -\int \frac{a}{\rho} d\rho \ + v \ ,$$

$$v = \frac{J_+ + J_-}{2}$$

$$a = a\left(J_+ - J_-\right)$$

Riemann invariant

= wave amplitude

Characteristics

= propergation speed

HD eq. in the conservation law. (1)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \,,$$

OK

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0,$$

Momentum

$$T\left(\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial x}\right) = 0 \,.$$

Energy

HD eq. in the conservation law. (2)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho v\right) = 0\,, \qquad (3)$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0, \quad (4)$$

$$\Downarrow \left(3\right)\times v + \left(4\right)\times \rho$$

$$\frac{\partial}{\partial t}\left(\rho v\right) + \frac{\partial}{\partial x}\left(\rho v^2 + P\right) = 0\,. \quad \text{Momentum}$$

HD eq. in the conservation law. (3)

$$T\left(\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial x}\right) = 0\,, \quad \Downarrow$$

Energy

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{v^2}{2} + \varepsilon\right)\right] + \frac{\partial}{\partial x}\left[\rho\left(\frac{v^2}{2} + \varepsilon + \frac{P}{\rho}\right)v\right] = 0\,,$$

Hints

$$T\,ds = dU - P\,dV = d(\rho\varepsilon) + \frac{P}{\rho^2}\,d\rho\,,$$

$$\left(\frac{v^2}{2} + \varepsilon\right)\frac{\partial \rho}{\partial t} + \rho v\,\frac{\partial v}{\partial t} + \frac{\partial}{\partial t}(\rho\varepsilon) = \frac{\partial}{\partial t}\left[\rho\left(\frac{v^2}{2} + \varepsilon\right)\right].$$

HD eq. in the conservation law. (4)

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho v \\ \rho E \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho v \\ \rho v^2 + P \\ \rho H v \end{bmatrix} = 0\,,$$

$$E = \frac{v^2}{2} + \varepsilon\,, \quad H = E + \frac{P}{\rho}\,.$$

HD eq. in the conservation law. (5)

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \,,$$

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho v \\ \rho E \end{bmatrix} , \quad \mathbf{F} = \begin{bmatrix} \rho v \\ \rho v^2 + P \\ \rho H v \end{bmatrix}$$

cf. The Burgers Eq.

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \,, \quad f = \frac{u^2}{2}$$

Review of the Burgers Eq.

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 , \quad f = \frac{u^2}{2} .$$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 , \quad c = \frac{df}{du} = u .$$

$$u_{j,n+1} = u_{j,n} - \frac{\Delta t}{\Delta x}\left(f_{j+1/2,n} - f_{j-1/2,n}\right)$$

$$f_{j+1/2} = \frac{1}{2}\left(f_{j+1,n} + f_{j,n}\right) - \frac{1}{2}\left|c_{j+1/2}\right|\left(u_{j+1,n} - u_{j,n}\right)$$

$$c_{j+1/2,n} = \frac{1}{2}\left(u_{j+1,n} + u_{j,n}\right)$$

Upwind Scheme for HD Eq. (1)

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0\,, \quad \mathbf{F} = \mathbf{F}(\mathbf{U})\,,$$

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} = 0, \quad \mathbf{A} \equiv \left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\right)$$

$$\mathbf{U}_{j,n+1} = \mathbf{U}_{j,n} - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{j+1/2,n} - \mathbf{F}_{j-1/2,n}\right),$$

$$\mathbf{F}_{j+1/2} = \frac{1}{2}\left(\mathbf{F}_{j+1,n} + \mathbf{F}_{j,n}\right) - \frac{1}{2}\left|\mathbf{A}_{j+1/2}\right|\left(\mathbf{U}_{j+1,n} - \mathbf{U}_{j,n}\right).$$

$|\mathbf{A}_{\mathbf{j+1/2}}|$ Average Absolute Velocity Matrix??

Upwind Scheme for HD Eq. (2)

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho v \\ \rho E \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho v \\ \rho v^2 + P \\ \rho H v \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix},$$

$$\mathbf{A} \equiv \left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}} \right)$$

$$F_1 = U_2, \quad F_2 = \frac{(U_2)^2}{U_1} + P$$

$$\varepsilon = \frac{1}{\gamma - 1} \frac{N_A k T}{\mu} = E - \frac{v^2}{2},$$

$$P = \frac{N_A k \rho T}{\mu} = (\gamma - 1)\left[U_3 - \frac{1}{2}\left(\frac{U_2}{U_1}\right)^2 \right]$$

Upwind Scheme for HD Eq. (3)

$$\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} U_2 \\ \dfrac{3-\gamma}{2}\dfrac{(U_2)^2}{U_1} \\ \dfrac{U_2}{U_1}\left(\gamma U_3 - (\gamma - 1)\dfrac{(U_2)^2}{2U_1}\right) \end{bmatrix}$$

$$A_{11} = \frac{\partial F_1}{\partial U_1} = 0\,, \quad A_{12} = \frac{\partial F_1}{\partial U_2} = 1\,, \ldots..$$

Textbook

(1. 79)

Upwind Scheme for HD Eq. (4)

How do we get the absolute value of the matrix, A?

diagonalize

$$\mathbf{A} = \mathbf{R} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \mathbf{L} ,$$

$$\mathbf{L}\,\mathbf{R} = \mathbf{R}\,\mathbf{L} = 1 , \quad \mathbf{R} = \left(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \right) ,$$

$$\mathbf{A}\,\mathbf{r}_i = \lambda_i\,\mathbf{r}_i$$

Eigenvalues and Eigenvectors

Upwind Scheme for HD Eq. (4)

How do we get the absolute value of the matrix, A?

diagonalize

$$|\mathbf{A}| \neq \begin{pmatrix} |A_{11}| & |A_{12}| & |A_{13}| \\ |A_{21}| & |A_{22}| & |A_{23}| \\ |A_{31}| & |A_{32}| & |A_{33}| \end{pmatrix}$$

Eigenvalues and Eigenvectors

Upwind Scheme for HD Eq. (5)

$$\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0 \;\Rightarrow$$

$$\lambda_1 = v - c,\;\; \lambda_2 = v,\;\; \lambda_3 = v + a$$

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ v - a \\ H - a\,v \end{pmatrix},\;\; \mathbf{r}_2 = \begin{pmatrix} 1 \\ v \\ \dfrac{v^2}{2} \end{pmatrix},\;\; \mathbf{r}_3 = \begin{pmatrix} 1 \\ v + a \\ H + a\,v \end{pmatrix}$$

Upwind Scheme for HD Eq. (5)

$$L = \frac{\gamma - 1}{2a^2} \begin{pmatrix} \frac{v^2}{2} + \frac{va}{\gamma - 1} & -v + \frac{a}{\gamma - 1} & 1 \\ -v^2 + \frac{2a^2}{\gamma - 1} & 2v & -2 \\ -\frac{v^2}{2} - \frac{va}{\gamma - 1} & v - \frac{a}{\gamma - 1} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{l}_1 \\ \mathbf{l}_2 \\ \mathbf{l}_3 \end{pmatrix}$$

$$\mathbf{l}_i \, \mathbf{A} = \lambda_i \, \mathbf{l}_i \,, \qquad \mathbf{l}_i \bullet \mathbf{r}_j = \delta_{ij} \,, \quad \mathbf{LR} = 1$$

Upwind Scheme for HD Eq. (6)

Where do we evaluate $\mathbf{A_{j+1/2}}$?

Roe average

$$\bar{v} = \frac{\sqrt{\rho_j}\, v_j + \sqrt{\rho_{j+1}}\, v_{j+1}}{\sqrt{\rho_j} + \sqrt{\rho_{j+1}}}$$

$$\bar{H} = \frac{\sqrt{\rho_j}\, H_j + \sqrt{\rho_{j+1}}\, H_{j+1}}{\sqrt{\rho_j} + \sqrt{\rho_{j+1}}}$$

$$\bar{a}^2 = (\gamma - 1)\left(\bar{H} - \frac{\bar{v}^2}{2}\right)$$

$$\bar{\rho} = \sqrt{\rho_{j+1}\, \rho_j}$$

Upwind Scheme for HD Eq. (7)

$$\mathbf{F}_{j+1} - \mathbf{F}_j = \overline{\mathbf{A}}\left(\mathbf{U}_{j+1} - \mathbf{U}_j\right)$$ ⬅ Property U

$$\mathbf{F}_{j+1/2} = \frac{1}{2}\left(\mathbf{F}_{j+1,n} + \mathbf{F}_{j,n} - \sum_{k=1}^{3} \left|\lambda_k\right| w_k \,\mathbf{r}_k\right)$$

$$w_k = \mathbf{l}_k . \bullet \left(\mathbf{U}_{j+1} - \mathbf{U}_j\right)$$

$$\mathbf{U}_{j+1} - \mathbf{U}_j = \sum_{k=1}^{3} w_k \,\mathbf{r}_k$$

$$\mathbf{F}_{j+1} - \mathbf{F}_j = \sum_{k=1}^{3} \lambda_k w_k \,\mathbf{r}_k$$

Upwind Scheme for HD Eq. (8)

Why ?

$$\mathbf{F}_{j+1/2} = \frac{1}{2}\left(\mathbf{F}_{j+1,n} + \mathbf{F}_{j,n}\right) - \frac{1}{2}\left|\mathbf{A}_{j+1/2}\right|\left(\mathbf{U}_{j+1,n} - \mathbf{U}_{j,n}\right).$$

When $v \geq a,\ \ |\lambda| = \lambda,\ |\mathbf{A}| = \mathbf{A}$, and $\mathbf{F}_{j+1/2} = \mathbf{F}_j$.

When $v \leq -a,\ \ |\lambda| = -\lambda,\ |\mathbf{A}| = -\mathbf{A}$, and $\mathbf{F}_{j+1/2} = \mathbf{F}_{j+1}$.

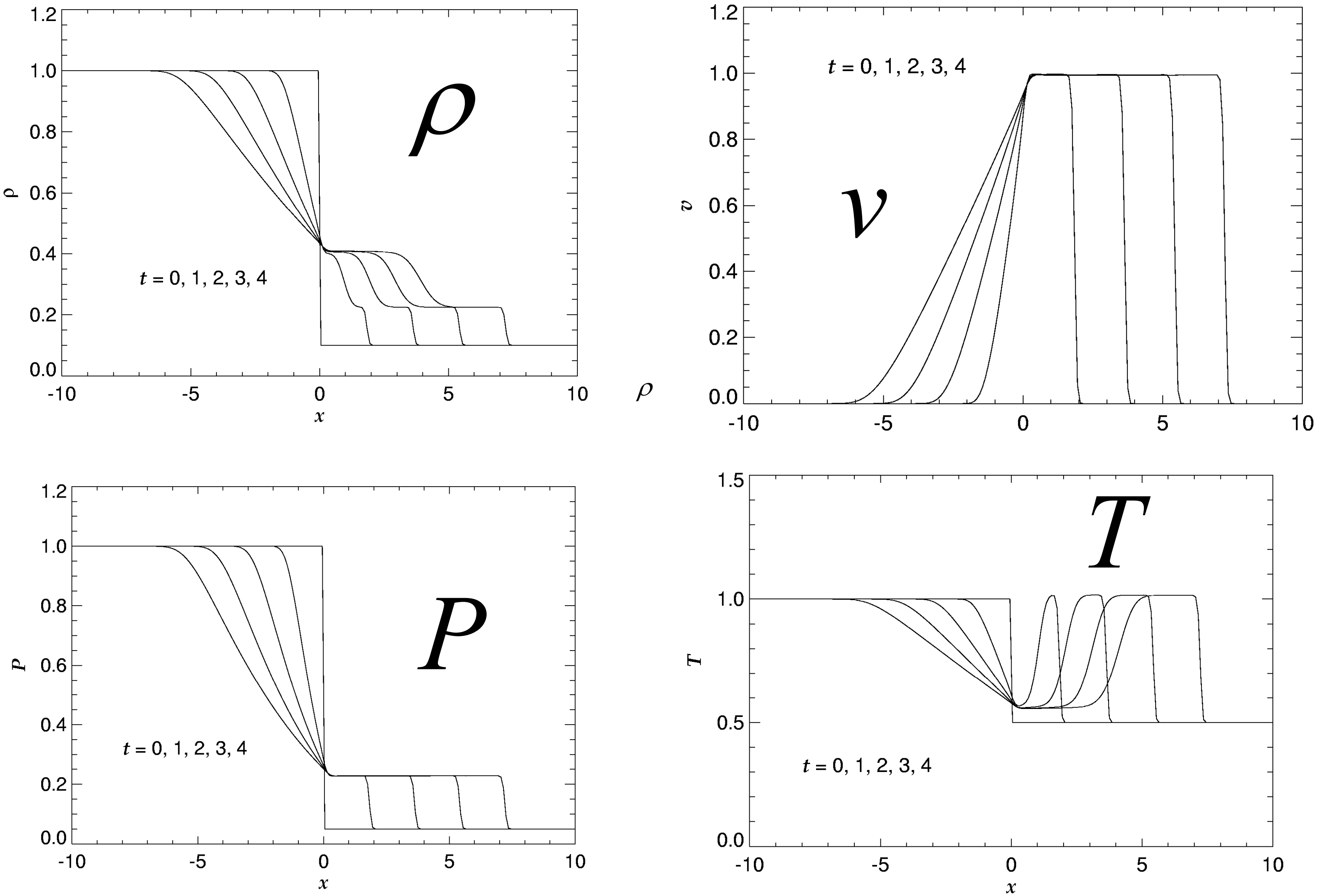
Otherwise, the numerical flux is mixture of $\boldsymbol{F}_{j+1}$ and $\boldsymbol{F}_j$. Flux difference is splitted into components in the Roe scheme.

Numerical Example: Shock Tube Problem

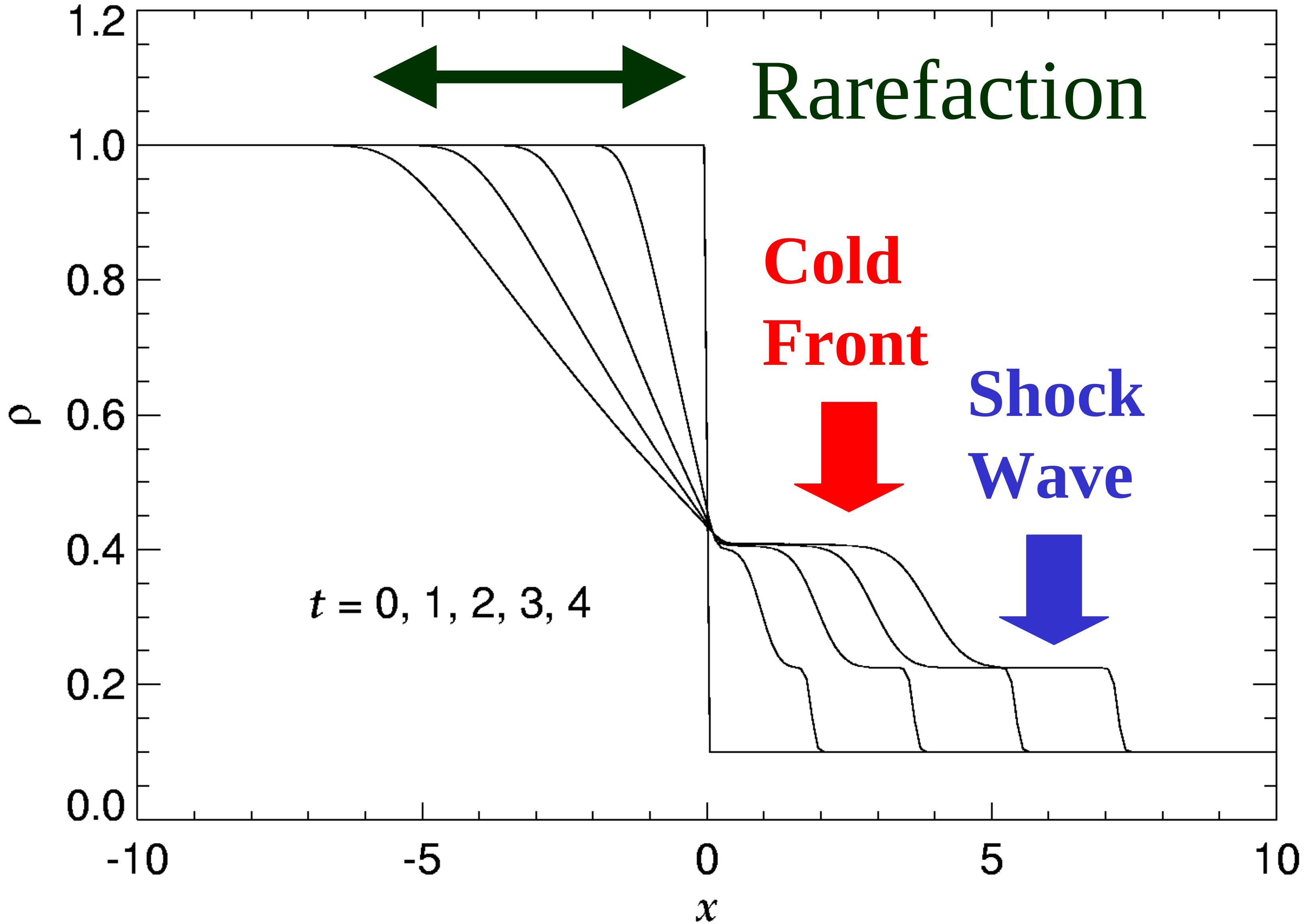
Initial Condition

	(left)	(right)
	$x \le 0$	$x > 0$
	$P = 1$	$P = 0.05$
	$\rho = 1$	$\rho = 0.1$
	$T = 1$	$T = 0.5$
	$v = 0$	$v = 0$

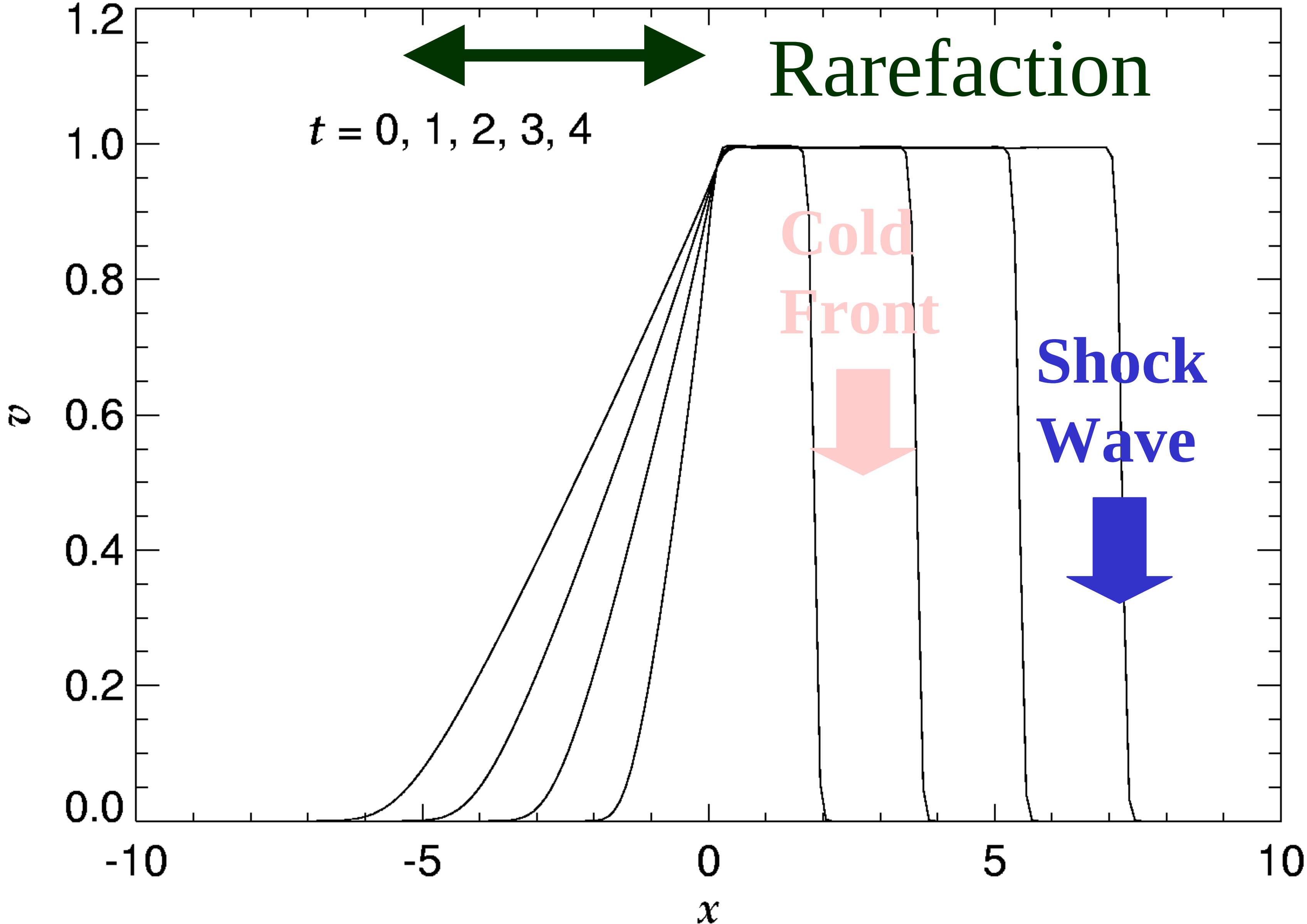
Shock Tube Problem (1)



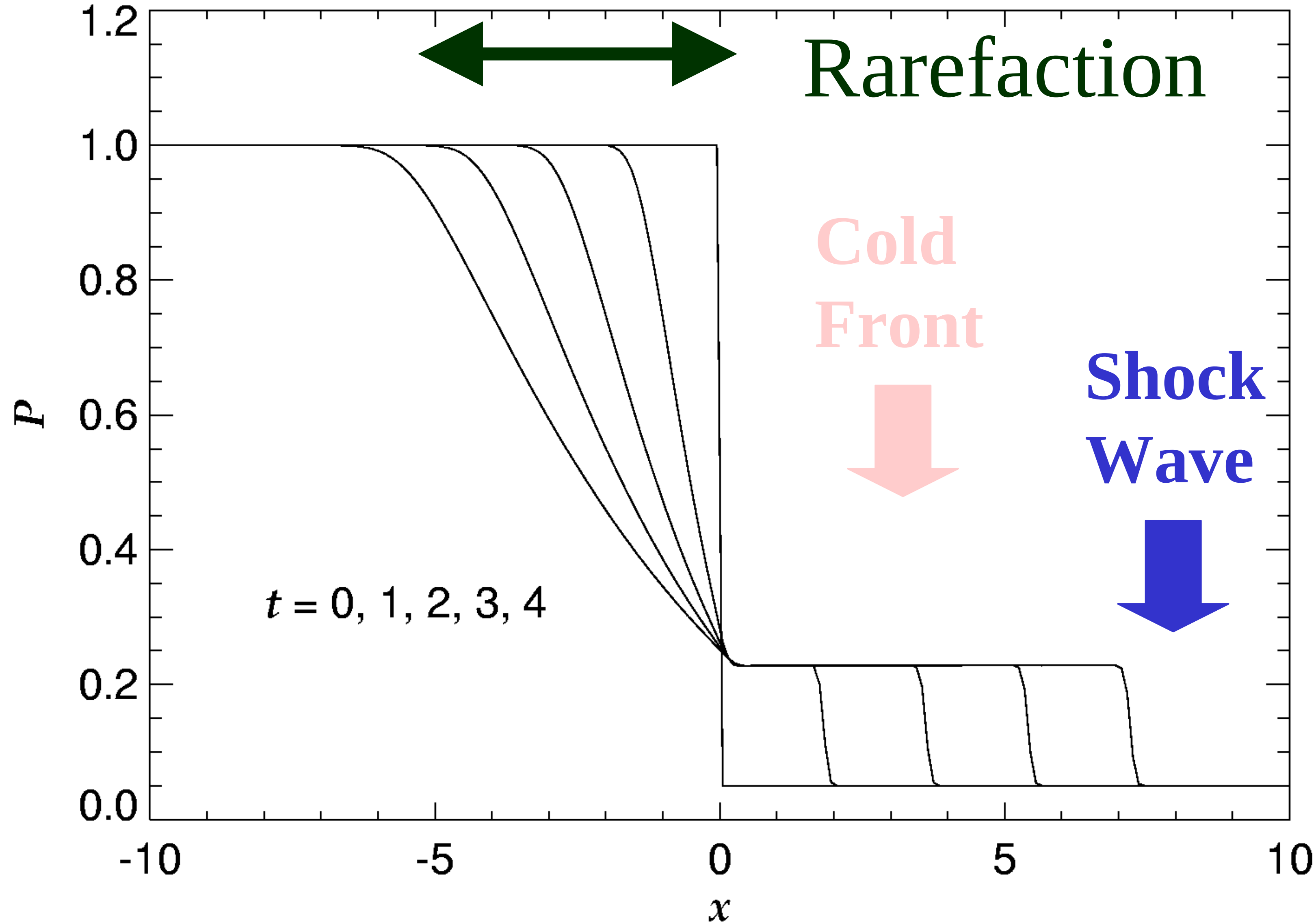
Shock Tube Problem (2)



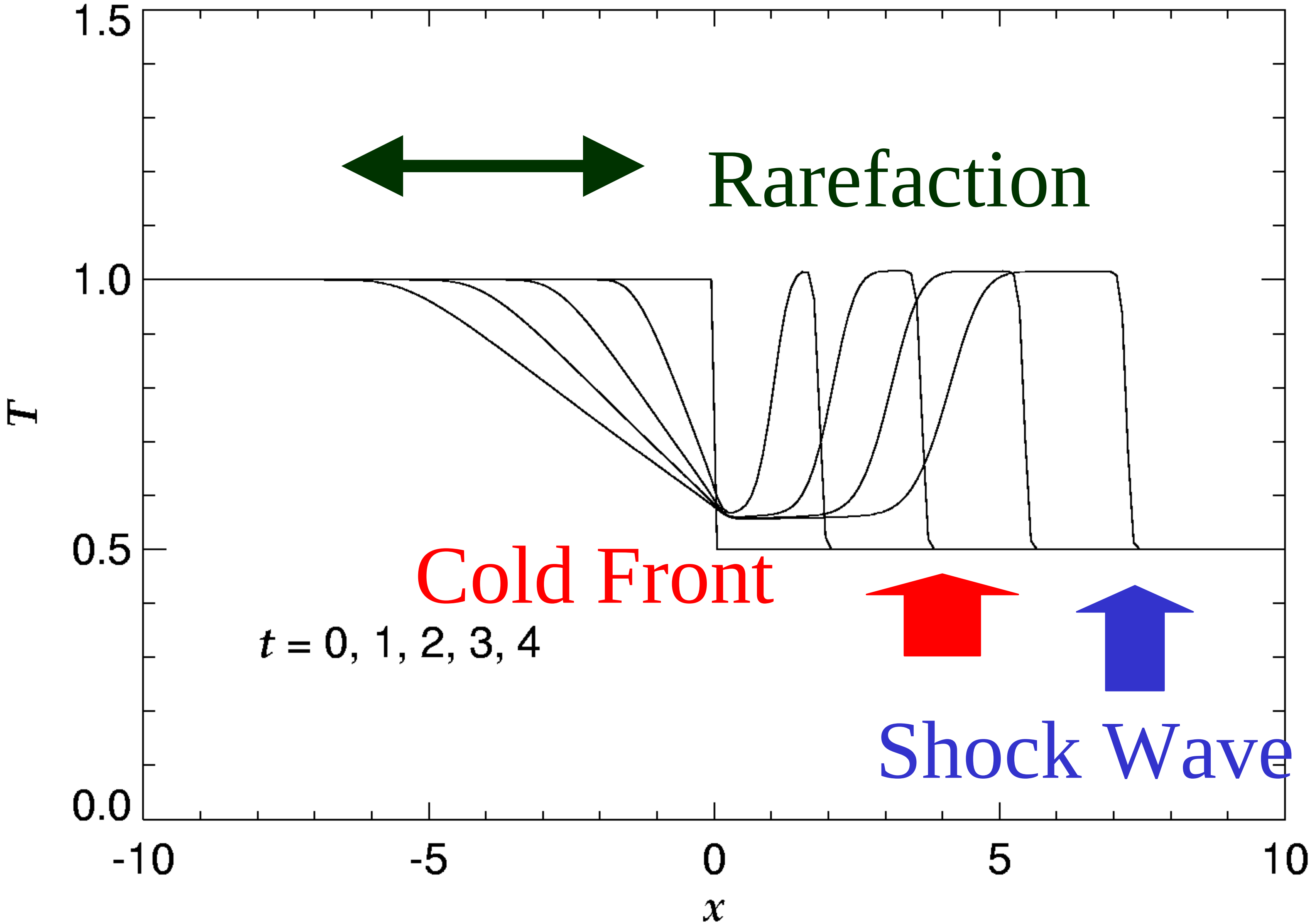
Shock Tube Problem (3)



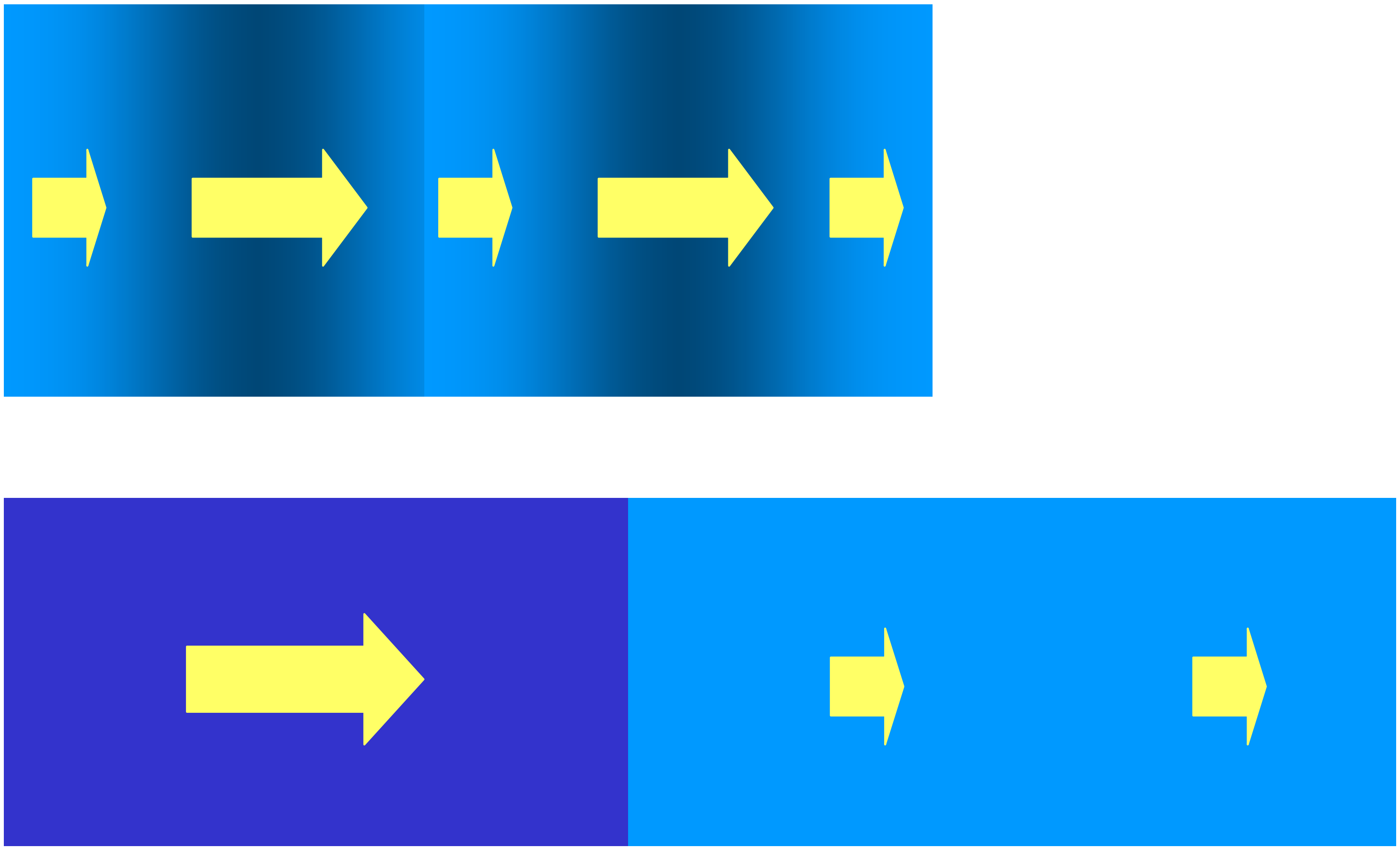
Shock Tube Problem (4)



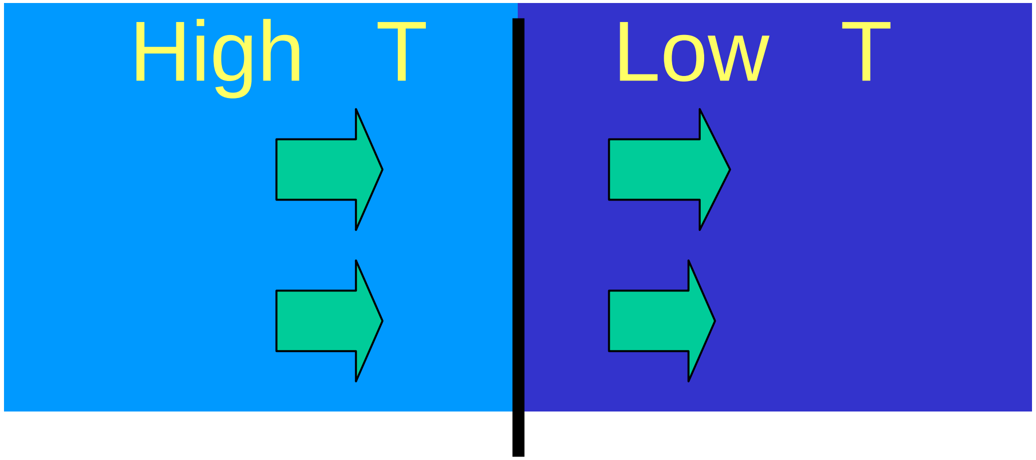
Shock Tube Problem (5)



Sound Wave and Shock Wave

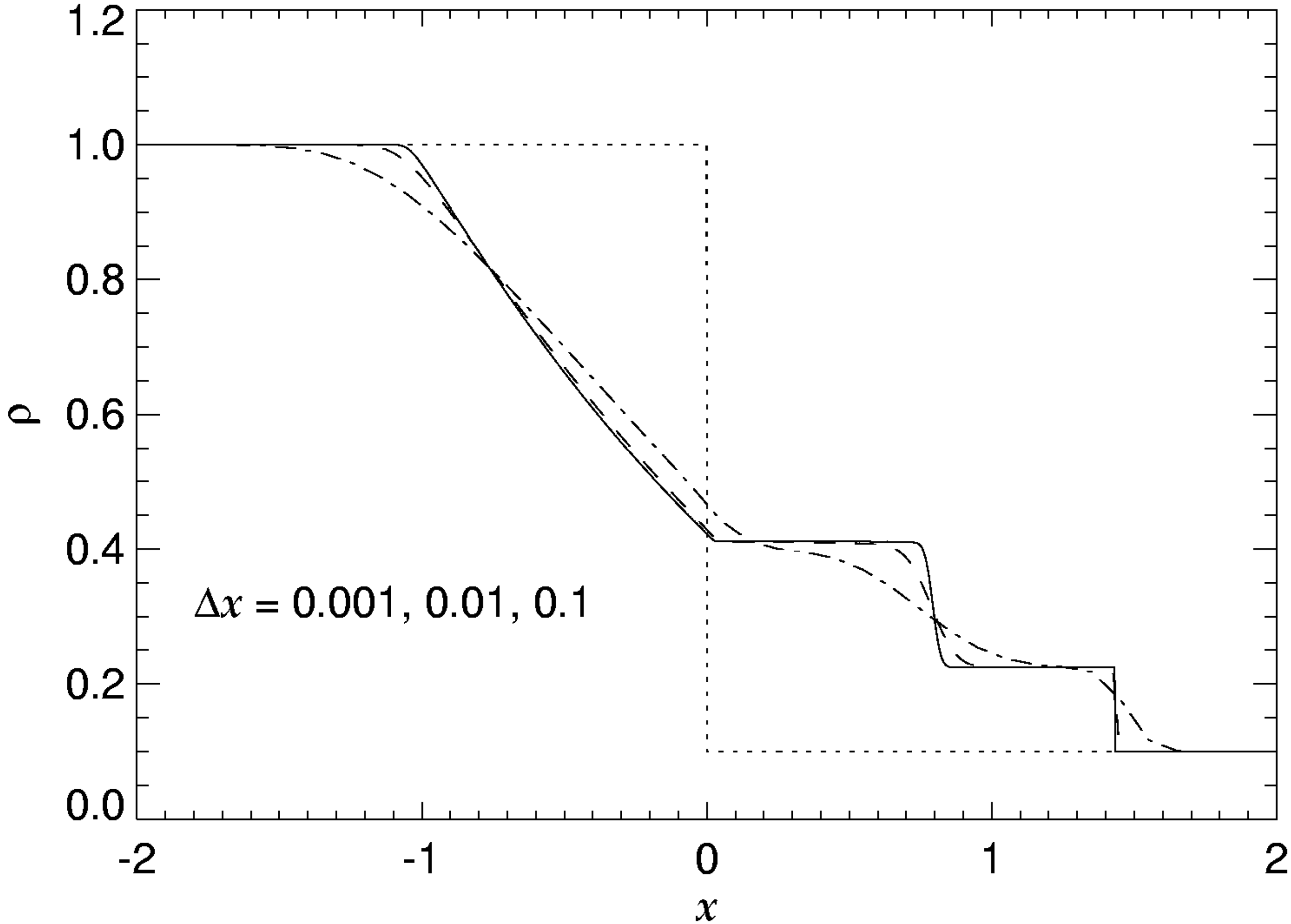


Entropy Wave

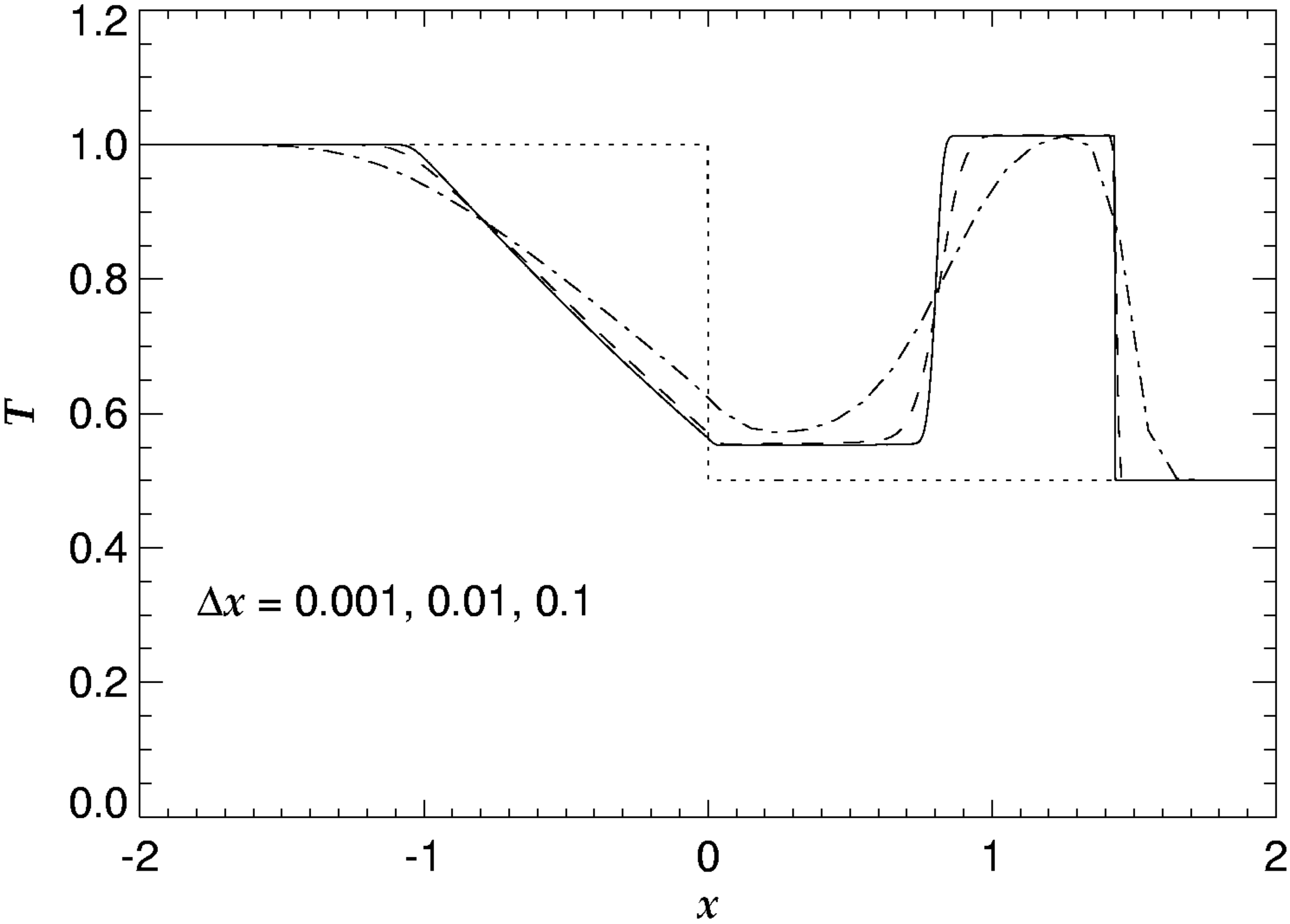


$$\rho_L \neq \rho_R \qquad P_L = P_R\,,\ u_L = u_R$$

Shock Tube Problem (6)



Shock Tube Problem (7)



Advanced Exercise for Roe Scheme

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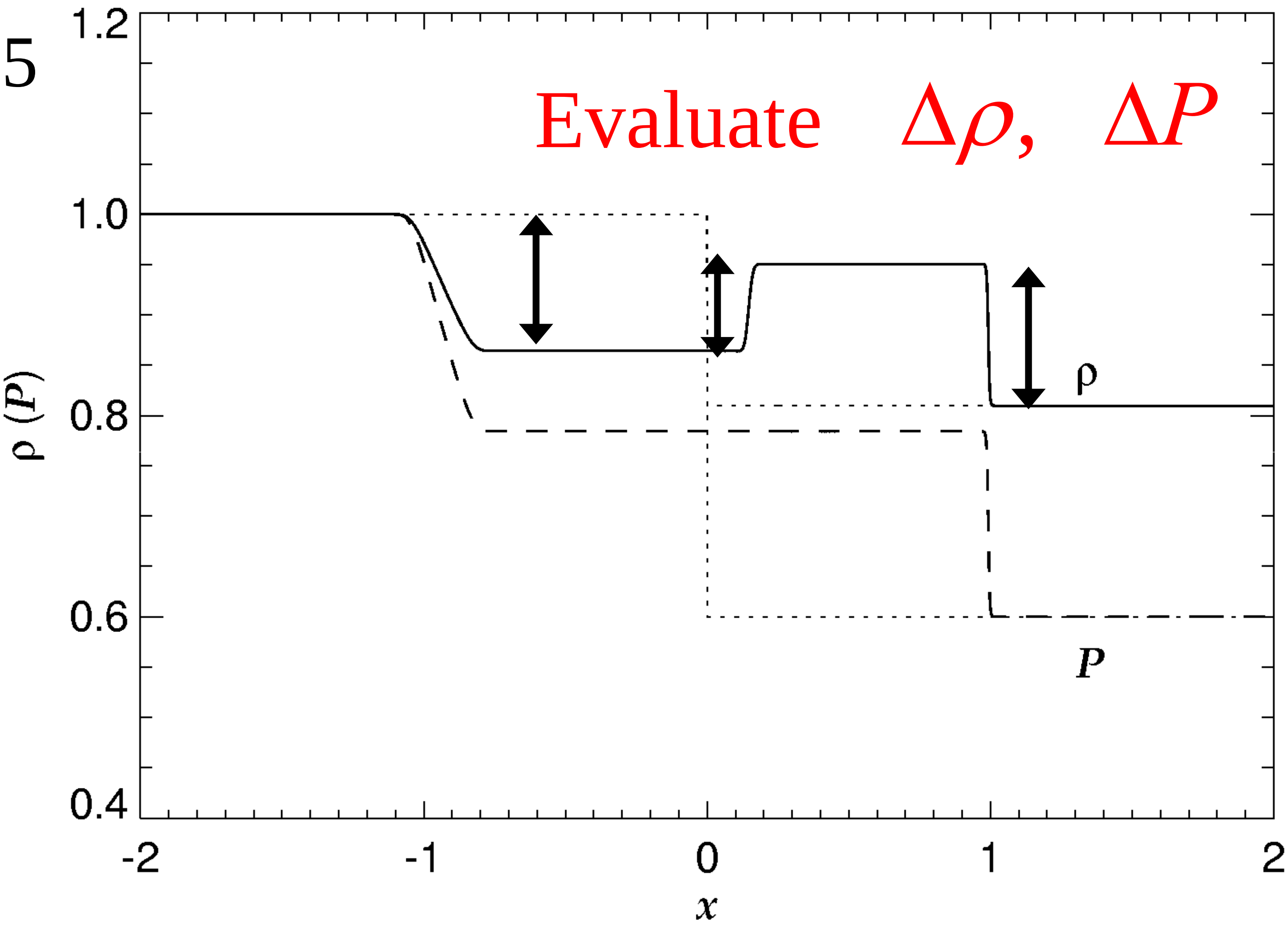

Riemann Invariants

$$d\mathbf{J} = d\mathbf{w} = L\, d\mathbf{U}$$

$$w_1 = -\frac{1}{2a}\left(\frac{\Delta P}{a} - \bar{\rho}\,\Delta v\right), \quad w_2 = \Delta\rho - \frac{\Delta P}{a^2}$$

$$w_3 = -\frac{1}{2a}\left(\frac{\Delta P}{a} - \bar{\rho}\,\Delta v\right)$$

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ v - a \\ H - a\,v \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ v \\ \dfrac{v^2}{2} \end{pmatrix}, \quad \mathbf{r}_3 = \begin{pmatrix} 1 \\ v + a \\ H + a\,v \end{pmatrix}$$

Advanced Exercise for Roe Scheme

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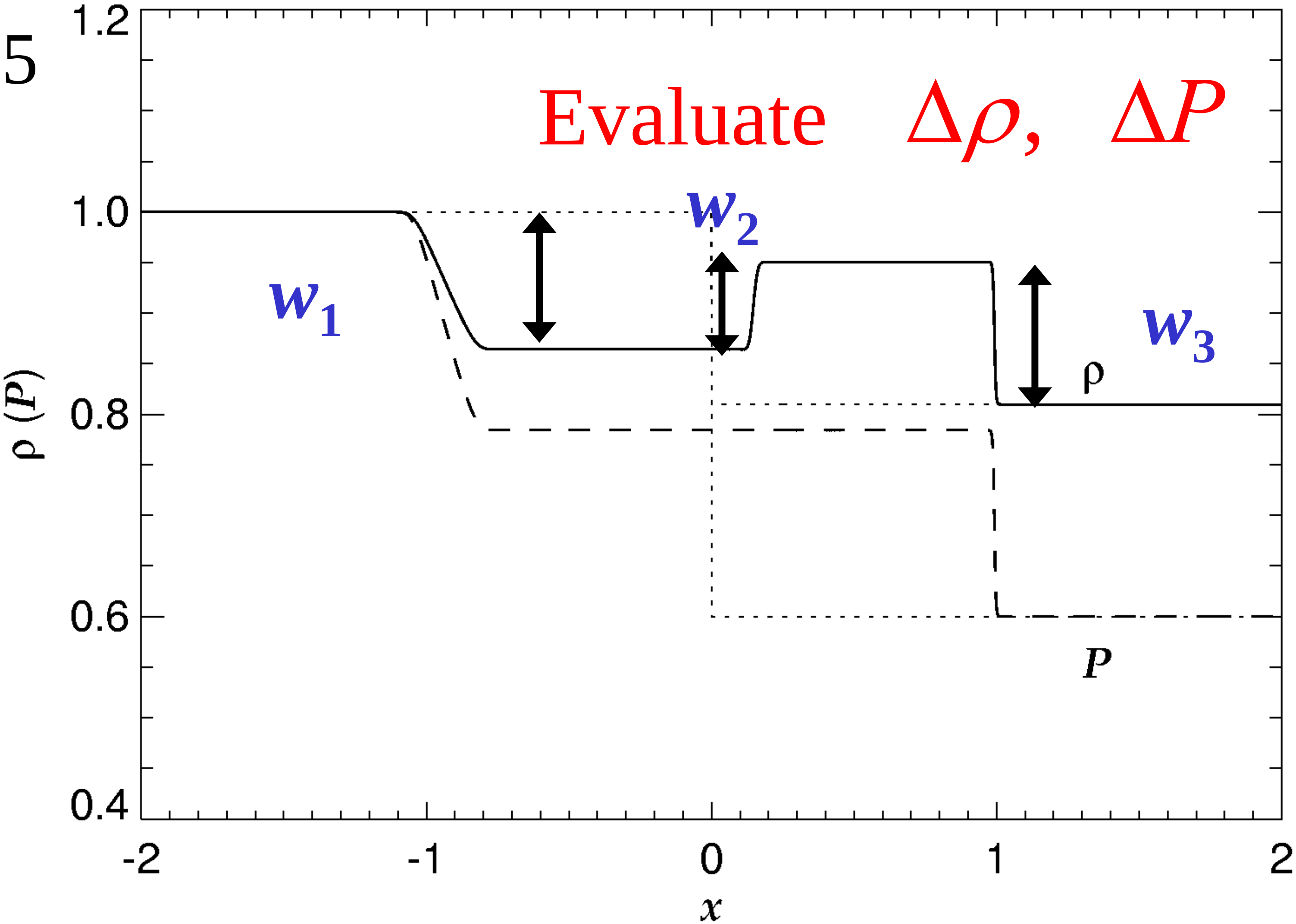

Another Example

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