Extension to Multi-dimensional Problems and MHD.

- 3D problem
- Gravity
- Heating & Cooling
- Cylindrical Coordinates
- MHD

Upwind Scheme

(yestedays' review)

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 , \quad \mathbf{F} = \mathbf{F}\left(\mathbf{U}\right),$$

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} = 0, \quad \mathbf{A} \equiv \left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\right)$$

$$\mathbf{U}_{j,n+1} = \mathbf{U}_{j,n} - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{j+1/2,n} - \mathbf{F}_{j-1/2,n}\right),$$

$$\mathbf{F}_{j+1/2} = \frac{1}{2}\left(\mathbf{F}_{j+1,n} + \mathbf{F}_{j,n}\right) - \frac{1}{2}\left|\mathbf{A}_{j+1/2}\right| \left(\mathbf{U}_{j+1,n} - \mathbf{U}_{j,n}\right).$$

3-D Problem (1)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 , \quad \Downarrow$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0,$$

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 , \quad \Downarrow$$

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} + \frac{\partial \mathbf{H}}{\partial z} = 0$$

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho v \\ \rho E \end{pmatrix} \Rightarrow \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix}$$

3-D Problem (2)

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} \rho u \\ \rho u^2 + P \\ \rho uv \\ \rho uw \\ \rho Hu \end{pmatrix}, \mathbf{G} = \begin{pmatrix} \rho v \\ \rho vu \\ \rho v^2 + P \\ \rho vw \\ \rho Hv \end{pmatrix}, \mathbf{H} = \begin{pmatrix} \rho w \\ \rho wu \\ \rho wv \\ \rho w^2 + P \\ \rho Hw \end{pmatrix}$$

Flux in x, y, and z.

Cyclic change

$$x \to y \to z$$

$$u \to v \to w$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \to \begin{pmatrix} 1 \\ 4 \\ 2 \\ 3 \\ 5 \end{pmatrix} \to \begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \\ 5 \end{pmatrix}$$

$$\mathbf{F} \to \mathbf{G} \to \mathbf{H}$$

3D Problem (3)

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \rho u \\ \rho u^2 + P \\ \rho uv \\ \rho uw \\ \rho Hu \end{pmatrix}$$

5 comp.

$$A = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} : \; 5 \times 5 \, , \quad \lambda_k \, , \; \mathbf{r}_k \, , \; k = 1, \, 2, 3, 4, \, 5$$

pp. 43- 44

3D Problem (4)

$$\lambda_1 = u - a,$$

$$\lambda_2 = \lambda_3 = \lambda_4 = u,$$

$$\lambda_1 = u + a,$$

$$\mathbf{r}_{5,1} = \begin{pmatrix} 1 \\ u \pm a \\ v \\ w \\ H \pm ua \end{pmatrix}$$

New!

$$w_{5,1} = \frac{1}{2a}\left(\frac{\Delta P}{a} \pm \rho \Delta v\right)$$

Sound Waves

3D Problem (5)

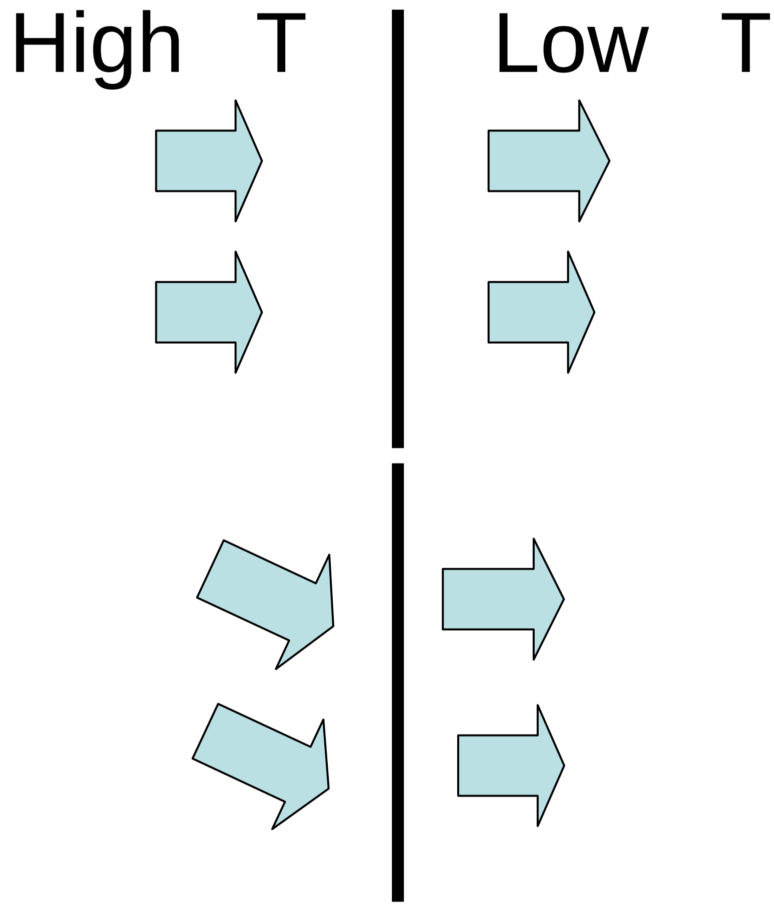
$$\lambda_2 = \lambda_3 = \lambda_4 = u$$

$$\mathbf{r}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -a \\ wa \end{pmatrix}, \ \mathbf{r}_3 = \begin{pmatrix} 0 \\ 0 \\ a \\ 0 \\ va \end{pmatrix}, \ \mathbf{r}_4 = \begin{pmatrix} 1 \\ u \\ v \\ w \\ q \end{pmatrix}, \quad q = \frac{u^2 + v^2 + w^2}{2}.$$

Shear z, y entropy

3D Problem (6)

$$P_L = P_R\,,\ u_L = u_R$$



entropy wave

$\rho_L \neq \rho_R$

shear wave

$\rho_L = \rho_R$

3D Problem (7)

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} + \frac{\partial \mathbf{H}}{\partial z} = 0\,,$$

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{U}}{\partial y} + \mathbf{C}\frac{\partial \mathbf{U}}{\partial z} = 0,$$

$$\mathbf{A} \equiv \left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\right),\ \mathbf{B} \equiv \left(\frac{\partial \mathbf{G}}{\partial \mathbf{U}}\right), \mathbf{C} \equiv \left(\frac{\partial \mathbf{H}}{\partial \mathbf{U}}\right)$$

Three velocity Matrixes

3D Problem (8)

$$\mathbf{F}_{j+1/2} = \frac{1}{2}\left(\mathbf{F}_{j+1,n} + \mathbf{F}_{j,n} - \sum_{k=1}^{5} \left|\lambda_k\right| w_k \, \mathbf{r}_k \right)$$

$$w_k = \mathbf{l}_k . \bullet \left(\mathbf{U}_{j+1} - \mathbf{U}_j \right)$$

$$\mathbf{U}_{j+1} - \mathbf{U}_j = \sum_{k=1}^{5} w_k \, \mathbf{r}_k$$

$$\mathbf{F}_{j+1} - \mathbf{F}_j = \sum_{k=1}^{5} \lambda_k \, w_k \, \mathbf{r}_k$$

3D Problem (9)

$$\mathbf{U}_{j,n+1} = \mathbf{U}_{j,n} - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{j+1/2,n} - \mathbf{F}_{j-1/2,n}\right)$$

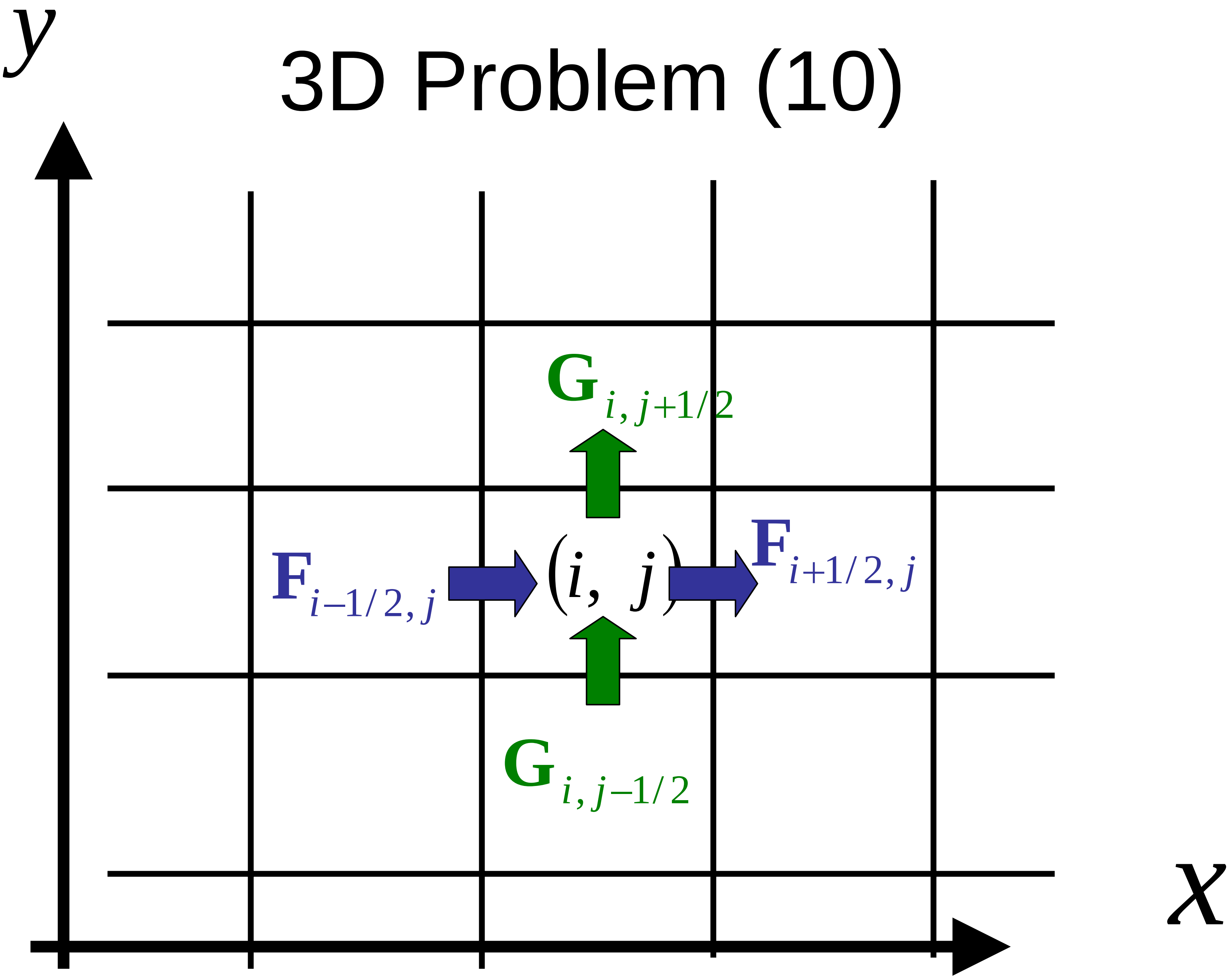
$$- \frac{\Delta t}{\Delta y}\left(\mathbf{G}_{j+1/2,n} - \mathbf{G}_{j-1/2,n}\right) - \frac{\Delta t}{\Delta z}\left(\mathbf{H}_{j+1/2,n} - \mathbf{H}_{j-1/2,n}\right),$$

$$\mathbf{F}_{j+1/2} = \frac{1}{2}\left(\mathbf{F}_{j+1,n} + \mathbf{F}_{j,n}\right) - \frac{1}{2}\left|\mathbf{A}_{j+1/2}\right|\left(\mathbf{U}_{j+1,n} - \mathbf{U}_{j,n}\right),$$

$$\mathbf{G}_{j+1/2} = \frac{1}{2}\left(\mathbf{G}_{j+1,n} + \mathbf{G}_{j,n}\right) - \frac{1}{2}\left|\mathbf{B}_{j+1/2}\right|\left(\mathbf{U}_{j+1,n} - \mathbf{U}_{j,n}\right),$$

$$\mathbf{H}_{j+1/2} = \frac{1}{2}\left(\mathbf{H}_{j+1,n} + \mathbf{H}_{j,n}\right) - \frac{1}{2}\left|\mathbf{C}_{j+1/2}\right|\left(\mathbf{U}_{j+1,n} - \mathbf{U}_{j,n}\right)$$

3D Problem (10)



Gravity, Heating & Cooling (1)

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{v}) = 0\ ,$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \bullet \nabla)\mathbf{v} + \frac{1}{\rho}\nabla P = \mathbf{g}\,,$$

$$\rho\ T\frac{Ds}{Dt} = \Gamma - \Lambda\ .$$

$$\mathbf{g} = \mathbf{g}(\mathbf{r},\ \mathbf{t})$$

$$\Gamma = \Gamma(\rho, T)\ ,$$

$$\Lambda = \Lambda(\rho, T)\ .$$

Gravity, Heating & Cooling (2)

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} + \frac{\partial \mathbf{H}}{\partial z} = \mathbf{S} \,,$$

$$\mathbf{S} = \begin{pmatrix} 0 \\ \rho\,\mathbf{g} \\ \rho\,\mathbf{g} \bullet \mathbf{v} + \Lambda - \Gamma \end{pmatrix}.$$

Gravity, Heating & Cooling (3)

$$\mathbf{U}_{j,n+1} = \mathbf{U}_{j,n} - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{j+1/2,n} - \mathbf{F}_{j-1/2,n}\right)$$

$$- \frac{\Delta t}{\Delta y}\left(\mathbf{G}_{j+1/2,n} - \mathbf{G}_{j-1/2,n}\right) - \frac{\Delta t}{\Delta z}\left(\mathbf{H}_{j+1/2,n} - \mathbf{H}_{j-1/2,n}\right)$$

$$+ \Delta t \ \mathbf{S}_{j,n} ,$$

New!

Cylindrical Coordinate (1)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho v_r\right) + \frac{\partial}{\partial z}\left(\rho v_z\right) = 0 \text{ , } \Downarrow \times r$$

$$\frac{\partial}{\partial t}\left(r\rho\right) + \frac{\partial}{\partial r}\left(r\rho v_r\right) + \frac{\partial}{\partial z}\left(r\rho v_z\right) = 0.$$

$$\int \quad dV = \int \quad \boxed{r\ dr}\, d\phi\, dz$$

Finite Element (Volume) method

Cylindrical Coordinates (2)

$$\frac{\partial}{\partial t}\left(r\ \mathbf{U}\right) + \frac{\partial}{\partial t}\left(r\ \mathbf{F}_r\right) + \frac{\partial}{\partial t}\left(r\ \mathbf{F}_z\right) = \mathbf{S}$$

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho v_r \\ \rho v_z \\ \rho E \end{pmatrix},\ \mathbf{F}_r = \begin{pmatrix} \rho\ v_r \\ \rho v_r{}^2 + P \\ \rho v_r\, v_z \\ \rho H\, v_r \end{pmatrix},\ \mathbf{F}_z = \begin{pmatrix} \rho\ v_z \\ \rho v_z v_r \\ \rho v_z{}^2 + P \\ \rho H\, v_z \end{pmatrix},\ \mathbf{S} = \begin{pmatrix} 0 \\ P \\ 0 \\ 0 \end{pmatrix}$$

$$r\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left(rP\right) - P \qquad r_{j+1/2} = r_j \text{ or } r_{j+1}???$$

Cylindrical Coordinates (3)

$$v_\varphi \Rightarrow r\, v_\varphi$$

$$\rho\, v_\varphi \Rightarrow (r\,\rho)\left(r\, v_\varphi\right) = r^{2}\,\rho\, v_\varphi$$

Angular Momentum Conservation

Extension to MHD Equations

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ B_x \\ B_y \\ B_z \\ \rho E \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \rho v_x \\ \rho v_x^2 + P + \dfrac{B_y^2 + B_z^2 - B_x^2}{8\pi} \\ \rho v_x v_y - \dfrac{B_x B_y}{4\pi} \\ \rho v_x v_z - \dfrac{B_x B_z}{4\pi} \\ 0 \\ v_x B_y - v_y B_x \\ v_x B_z - vz B_x \\ \rho H v_x - \dfrac{B_x \mathbf{B} \bullet \mathbf{v}}{4\pi} \end{pmatrix}$$

8 comp.

7 waves

$$\nabla \bullet B = 0$$

Fast $\times$ 2 + Slow $\times$ 2 + Alfven $\times$ 2 + Entropy