I. Finite difference methods for 1D scalar equation

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Differential Eq → Finite-Difference

$$\frac{\partial f}{\partial t}+c\frac{\partial f}{\partial x}=0$$ 1D linear scalar equation

given a function f(x) → calculate df/dx

Several expressions of finite-difference

$$\left.\frac{\partial f}{\partial x}\right|_x \simeq \frac{f(x+\Delta x)-f(x)}{\Delta x}\quad \Delta x\text{:small}$$

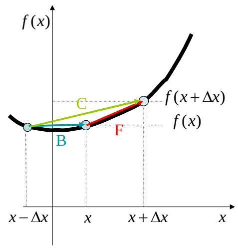
Forward difference

$$\left.\frac{\partial f}{\partial x}\right|_x \simeq \frac{f(x)-f(x-\Delta x)}{\Delta x}\quad \Delta x\text{:small}$$

Backward difference

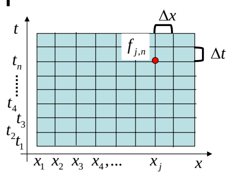
$$\left.\frac{\partial f}{\partial x}\right|_x \simeq \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}\quad \Delta x\text{:small}$$

Central difference



Grid in space

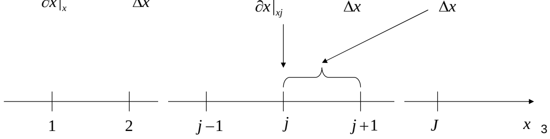
In the finite-difference method, space and time are divided into cells with a finite volume ($\Delta x \Delta t$).



Forward difference in space

$$\left.\frac{\partial f}{\partial x}\right|_x \simeq \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$\left.\frac{\partial f}{\partial x}\right|_{xj} \simeq \frac{f(x_{j+1})-f(x_j)}{\Delta x}=\frac{f_{j+1}-f_j}{\Delta x}$$

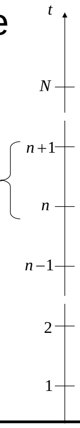


Grid in time

Forward difference in time

$$\left.\frac{\partial f_x}{\partial t}\right|_t \simeq \frac{f_x(t+\Delta t)-f_x(t)}{\Delta t}$$

$$\left.\frac{\partial f}{\partial t}\right|_{t_n} \simeq \frac{f_j(t_{n+1})-f_j(t_n)}{\Delta t}=\frac{f_{j,n+1}-f_{j,n}}{\Delta t}$$

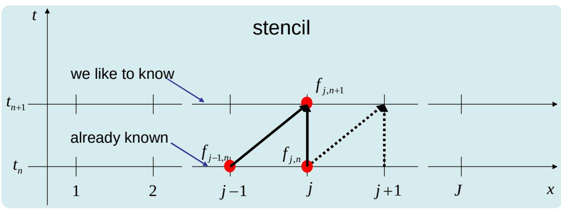


$$\left.\frac{\partial f}{\partial t}\right|_{t_n}$$

Scheme

Differential equation		Finite-difference equation
$\frac{\partial f}{\partial t}+c\frac{\partial f}{\partial x}=0$	⇨	$\frac{f_{j,n+1}-f_{j,n}}{\Delta t}+c\frac{f_{j,n}-f_{j-1,n}}{\Delta x}=0$

Forward difference in time and backward difference in space is used.



Flux

$$\frac{\partial f}{\partial t}+c\frac{\partial f}{\partial x}=0$$

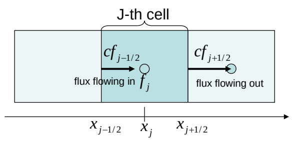
integrate

$$\int_{x_{j-1/2}}^{x_{j+1/2}}\frac{\partial f}{\partial t}dx+c\int_{x_{j-1/2}}^{x_{j+1/2}}\frac{\partial f}{\partial x}dx=0$$

$$\frac{\partial f_j}{\partial t}(x_{j+1/2}-x_{j-1/2})+\left[cf\right]_{x_{j-1/2}}^{x_{j+1/2}}=0$$

Change of quantity f

Difference in flux



How to derive the flux at the cell boundary → Scheme

Numerical Flux $cf_{j+1/2}$

This is not given a priori.

Calculation

- Incorrect but straight-forward code.

- Do j=1,N
-   f(j)=f(j)-cfl*(f(j)-f(j-1))
- End do

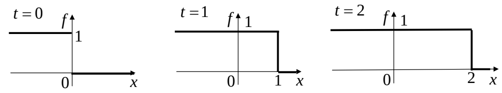
- Correct code

- Do j=1,N
-   F(j)=c*f(j-1)
- End do
- Do j=1,N
-   f(j)=f(j)-dt/dx*(F(j+1)-F(j))
- End do

Test Problem

$$\frac{\partial f}{\partial t}+c\frac{\partial f}{\partial x}=0;\quad c>0$$

$c=1$ Left-wind with a speed c=1



Eq(1.23)

Stability I. – Upwind --

$$\frac{f_{j,n+1}-f_{j,n}}{\Delta t}+\frac{F_{j+1/2}-F_{j-1/2}}{\Delta x}=0$$

Forward difference in space

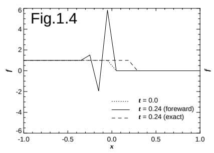
$$\frac{f_{j,n+1}-f_{j,n}}{\Delta t}+c\frac{f_{j+1,n}-f_{j,n}}{\Delta x}=0$$
$$F_{j+1/2}=c\,f_{j+1}$$

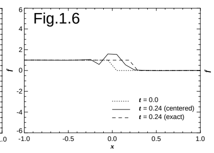
Central difference

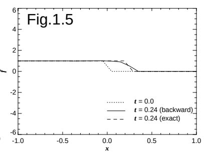
$$\frac{f_{j,n+1}-f_{j,n}}{\Delta t}+c\frac{f_{j+1,n}-f_{j-1,n}}{2\Delta x}=0$$
$$F_{j+1/2}=c(f_j+f_{j+1})/2$$

Backward difference

$$\frac{f_{j,n+1}-f_{j,n}}{\Delta t}+c\frac{f_{j,n}-f_{j-1,n}}{\Delta x}=0$$
$$F_{j+1/2}=c\,f_j$$







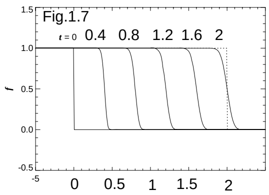
$\Delta x=0.1, \Delta t=0.08, c=1,$
CFL number $c\Delta t/\Delta x=0.8$

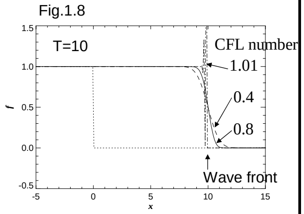
Stability II. -- CFL condition --

CFL number $=c\Delta t/\Delta x$

Backward difference

$$\frac{f_{j,n+1}-f_{j,n}}{\Delta t}+c\frac{f_{j,n}-f_{j-1,n}}{\Delta x}=0$$
$$F_{j+1/2}=c\,f_j$$





$\Delta x=0.1, \Delta t=0.08, c=1,$
CFL number $c\Delta t/\Delta x=0.8$

Even stable backward difference scheme,

For stability CFL # must be <1.
Smaller CFL # gives a more diffuse solution.

Errors in Finite-Difference

Taylor expansion $f(x-\Delta x)=f(x)-\left(\frac{\partial f}{\partial x}\right)_x\Delta x+\left(\frac{\partial^2 f}{\partial x^2}\right)_x\frac{\Delta x^2}{2!}-\left(\frac{\partial^3 f}{\partial x^3}\right)_x\frac{\Delta x^3}{3!}+O(\Delta x^4)$

$$\frac{f(x)-f(x-\Delta x)}{\Delta x}=\left(\frac{\partial f}{\partial x}\right)_x-\left(\frac{\partial^2 f}{\partial x^2}\right)_x\frac{\Delta x}{2!}+O(\Delta x^2)\quad(1.3)$$

Error proportional to Δx

$$\frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}=\left(\frac{\partial f}{\partial x}\right)_x+\left(\frac{\partial^3 f}{\partial x^3}\right)_x\frac{\Delta x^2}{3!}+O(\Delta x^3)\quad(1.4)$$

proportional to Δx^2

$$\frac{f(t+\Delta t)-f(t)}{\Delta t}=\left(\frac{\partial f}{\partial t}\right)_t+\left(\frac{\partial^2 f}{\partial t^2}\right)_t\frac{\Delta t}{2}+O(\Delta t^2)$$

proportional to Δt

$$=\left(\frac{\partial f}{\partial t}\right)_t+c^2\left(\frac{\partial^2 f}{\partial x^2}\right)_t\frac{\Delta t}{2}+O(\Delta t^2)$$

$$\frac{f(t+\Delta t)-f(t)}{\Delta t}+c\frac{f(x)-f(x-\Delta x)}{\Delta x}$$

(forward in time backward in space)

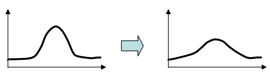
$$=\left(\frac{\partial f}{\partial t}\right)_t+c\left(\frac{\partial f}{\partial x}\right)_x+\underbrace{\left(\frac{\partial^2 f}{\partial x^2}\right)_t c\frac{c\Delta t-\Delta x}{2}}_{\text{Leading error}}+O(\Delta t^2)+cO(\Delta x^2)$$

Backward difference in space

$$\left(\frac{\partial f}{\partial t}\right)_t+c\left(\frac{\partial f}{\partial x}\right)_x=-\left(\frac{\partial^2 f}{\partial x^2}\right)_t c\frac{c\Delta t-\Delta x}{2}-O(\Delta t^2)-cO(\Delta x^2)$$

$c\Delta t<\Delta x\rightarrow$
$c>0$

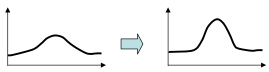
The leading term means a diffusion with positive diffusion constant



Forward difference in space

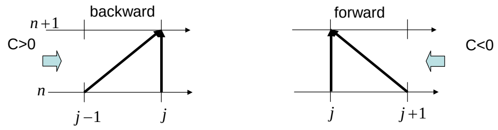
$$\left(\frac{\partial f}{\partial t}\right)_t+c\left(\frac{\partial f}{\partial x}\right)_x=-\left(\frac{\partial^2 f}{\partial x^2}\right)_t c\frac{c\Delta t+\Delta x}{2}-O(\Delta t^2)-cO(\Delta x^2)$$

$c>0\rightarrow$ The leading term means a diffusion with negative diffusion constant



Upwind

- For c>0, backward difference in space ➔ stable if CFL # <1
- For c<0, backward difference ➔ unstable
  forward difference ➔ stable
  (easily understood if you transform x →-x)
- This gives an idea that a stable finite difference must contain the quantity of UPWIND cells.



Expression of upwind numerical flux

$$F_{j+1/2} = c(f_j + f_{j+1})/2 - |c|(f_{j+1} - f_j)/2$$

Confirm this expression gives the following fluxes both upwind.

$c > 0$: $F_{j+1/2} = cf_j$

$c < 0$: $F_{j+1/2} = cf_{j+1}$

Stability

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = 0; \quad c > 0$$

If we know calculate

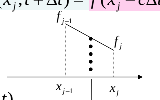
$$f(x_j, t+\Delta t) = f(x_j - c\Delta t, t)$$

--- Qualitative understanding ---

$\frac{f_{j,n+1} - f_{j,n}}{\Delta t} + c\frac{f_{j,n} - f_{j-1,n}}{\Delta x} = 0$ gives

Cfl #<1

$$f(x, t+\Delta t) = \left(1 - \frac{c\Delta t}{\Delta x}\right) f(x,t) + \frac{c\Delta t}{\Delta x} f(x - \Delta x, t) \quad (1.26)$$

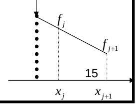


For c>0, this gives the interpolation between f(x,t) and f(x-Dx,t)

$\frac{f_{j,n+1} - f_{j,n}}{\Delta t} + c\frac{f_{j+1,n} - f_{j,n}}{\Delta x} = 0$ gives

$$f(x, t+\Delta t) = \left(1 + \frac{c\Delta t}{\Delta x}\right) f(x,t) - \frac{c\Delta t}{\Delta x} f(x + \Delta x, t) \quad (1.27)$$

c>0, the extrapolation from f(x,t) and f(x+Dx,t)



Von Neumann Analysis

Consider a plain wave

$f = \exp(ikx/\Delta x)$ kDx is an ordinary wavenumer.

x → x+Dx the phase of a wave increases at k. 0<k<π.

if we take $x_j = j\Delta x, f_{j,0} = \exp(ikj) \longrightarrow f_{j,n} = g^n \exp(ikj)$ (1.11)

g: complex amplification factor

lhs

$$\frac{f_{j,n+1} - f_{j,n}}{\Delta t} = \frac{g^{n+1} - g^n}{\Delta t}\exp(ikj) = g^n \frac{g-1}{\Delta t}\exp(ikj)$$

rhs of backward difference

amplification factor

$$c\frac{f_{j,n} - f_{j-1,n}}{\Delta x} = \frac{cg^n}{\Delta x}\left[\exp(ikj) - \exp[ik(j-1)]\right] = \frac{cg^n \exp(ikj)}{\Delta x}\left[1 - \exp[-ik]\right] \Rightarrow g = 1 - CFL\left[1 - \exp(-ik)\right] \quad (1.21)$$

rhs of forward difference

$$c\frac{f_{j,n} - f_{j-1,n}}{\Delta x} = \frac{cg^n}{\Delta x}\left[\exp[ik(j+1)] - \exp(ikj)\right] = \frac{cg^n \exp(ikj)}{\Delta x}\left[\exp[ik] - 1\right] \Rightarrow g = 1 - CFL\left[\exp(ik) - 1\right] \quad (1.20)$$

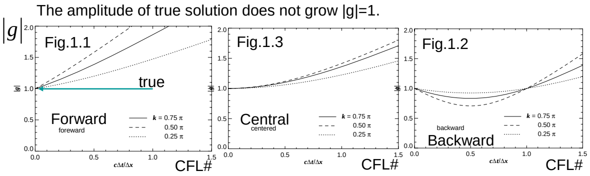
rhs of central difference

$$c\frac{f_{j,n} - f_{j-1,n}}{\Delta x} = \frac{cg^n}{\Delta x}\left[\exp[ik(j+1)] - \exp[ik(j-1)]\right] = \frac{cg^n \exp(ikj)}{\Delta x} i \sin k \Rightarrow g = 1 - iCFL\sin(k) \quad (1.22)$$

$$g = |g|\exp(i\phi) \qquad g_{exact} = \exp(-ikc\Delta t/\Delta x)$$

Error in amplitude

The amplitude of true solution does not grow |g|=1.



Forward and central difference in space give |g|>1.

|g(k)| >1 means such a wave grows.

Backward difference in space give |g|<1 for CFL # <1.

|g(k)| <1 means such a wave decays or amplitude of the wave decreases.

To solve the advection equation stably, |g(k)| <1 must be satisfied.

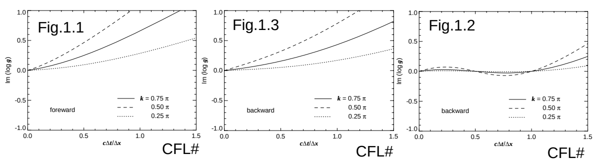
(1) suitable scheme + (2) small CFL #

Phase error

$$g = |g|\exp(i\phi) \qquad g_{exact} = \exp(-ikc\Delta t/\Delta x)$$

$$g/g_{exact} = |g|\exp(i(\phi + kc\Delta t/\Delta x))$$

Error in phase



CFL condition cΔt/Δx < 1

Summary of linear wave

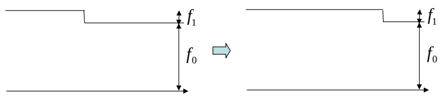
- For a finite-difference scheme to obtain stability,
- Upwind differentiation
- Satisfy CFL condition

Non-linear equation

Burger's equation $\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x} = 0$ (1.31)

Slightly nonlinear $f = f_0 + f_1(x,t)$ (1.32) $|f_0| \gg |f_1(x,t)|$

$$\frac{\partial(\cancel{f_0} + f_1)}{\partial t} + (f_0 + \cancel{f_1})\frac{\partial(\cancel{f_0} + f_1)}{\partial x} = 0 \Rightarrow \frac{\partial f_1}{\partial t} + f_0\frac{\partial f}{\partial x} = 0 \quad (1.33)$$



A part with larger amplitude propagates faster.

Burgers equation

$\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x} = 0 \Rightarrow$ Flux $F = \frac{f^2}{2}$

Upwind difference $\Rightarrow$

$$\left.\begin{array}{l} f_j > 0 : F_{j+1/2} = \frac{f_j^2}{2} \\ f_j < 0 : F_{j+1/2} = \frac{f_{j+1}^2}{2} \end{array}\right\} \frac{f_{j,n+1} - f_{j,n}}{\Delta t} + \frac{F_{j+1/2,n} - F_{j+1/2,n}}{\Delta x} = 0$$

or

$$F_{j+1/2} = \frac{1}{2}\left(\frac{f_{j+1}^2}{2} + \frac{f_j^2}{2}\right) - \frac{1}{2}\left|f_{j+1,n} + f_{j,n}\right|\left(f_{j+1,n} - f_{j,n}\right) \quad (1.45)$$

Initial condition $t = 0$ $\qquad$ $0 < t < 1$

$$f = \begin{cases} 1 & (x \le -1) \\ -x & (-1 < x \le 1) \\ -1 & x \ge 1 \end{cases} \Rightarrow f = \begin{cases} 1 & (x \le -1 + t) \\ -\frac{x}{t-1} & (-1 + t < x \le 1 - t) \\ -1 & (x \ge 1 - t) \end{cases}$$

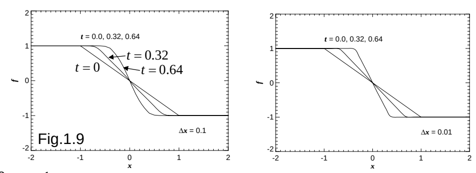


Fig.1.9

$0 < t < 1$

Solution steepens.

$t = 1$

Finally the solution has a sharp jump at x=0.

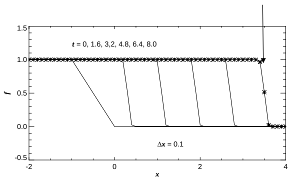
Negative gradient ➔ jump formation

$t \ge 1$

$$f = \begin{cases} 1 & (x < 0) \\ -1 & (x > 0) \end{cases}$$

Negative gradient ➔ jump formation

A jump similar to hydrodynamical shock front



Burgers (non-linear) equation predicts a jump, which corresponds to a shock front in the hydrodynamics.

$t = 0$ $\qquad$ $0 < t < 1$ $\qquad$ $t > 1$

$$f = \begin{cases} 1 & (x \le -1) \\ -x & (-1 < x \le 0) \\ 0 & x > 0 \end{cases} \Rightarrow f = \begin{cases} 1 & (x \le -1 + t) \\ -\frac{x}{1-t} & (-1 + t < x \le 0) \\ 0 & (x > 0) \end{cases} \Rightarrow f = \begin{cases} 1 & x \le t/2 - 1/2 \\ 0 & x > t/2 - 1/2 \end{cases}$$

Positive gradient

$$f = \begin{cases} -1 & (x \le -1) \\ x & (-1 < x \le 1) \\ 1 & x > 1 \end{cases}$$

Gradient is getting small in the region $-(1+t) < x < 1+t$

Gradient keeps its initial slope.

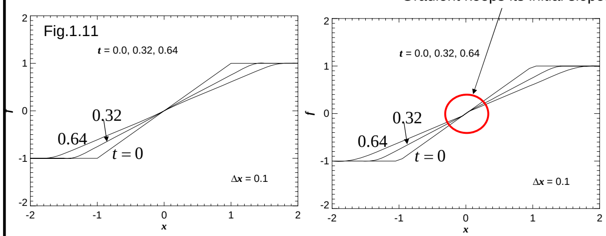


Fig.1.11

Near the point $x \simeq 0$, gradient keeps its initial slope.
Such a false structure is sometimes called an "expansion shock."
Sometimes it appears near the point where the direction of upwind changes.
In this model, x<0 right-wind and x>0 left-wind.

Entropy fix

$$F^*_{j+1/2} = \frac{1}{2}\left(\frac{f^2_{j+1}}{2} + \frac{f^2_j}{2}\right) - \frac{|\lambda|}{2}\left(f_{j+1,n} - f_{j,n}\right)$$

$$|\lambda| = \begin{cases} \dfrac{|f_{j+1,n} + f_{j,n}|}{2} & \left(\dfrac{|f_{j+1,n} + f_{j,n}|}{2} \geq \varepsilon\right) \\ \varepsilon & (\text{otherwise}) \end{cases}$$

$$\varepsilon = \max\left(0, \frac{f_{j+1,n} - f_{j,n}}{2}\right)$$

Fig.1.12