4. Higher-order Accuracy

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Upwind Flux (1)

For left-wind (c>0)

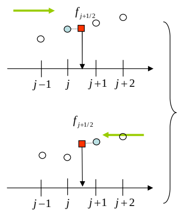
$$F^{(L)}_{j+1/2} = cf_j$$

$$f_{j+1/2} = f_j$$

For right-wind (c<0)

$$F^{(R)}_{j+1/2} = cf_{j+1}$$

$$f_{j+1/2} = f_{j+1}$$



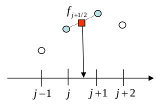
➔ upwind / 1st order

Upwind Flux (2)

To solve stably, finite-difference should take a manner of upwind into account.

Central difference / 2nd order

$$F_{j+1/2} = \frac{cf_j + cf_{j+1}}{2} \qquad f_{j+1/2} = \frac{f_j + f_{j+1}}{2}$$



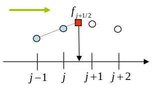
Upwind / 2nd order

For left-wind (c>0)

$$F^{(L)}_{j+1/2} = c\left(f_j + \frac{f_j - f_{j-1}}{2}\right) = c\frac{3f_j - f_{j-1}}{2}$$

Extrapolation j & j-1

$$f^{(L)}_{j+1/2} = \frac{3f_j - f_{j-1}}{2} = \frac{3f_j - \left[f_j - \Delta x\left(\frac{\partial f}{\partial x}\right)_{x_j} + O\left(\Delta x^2\right)\right]}{2} = f_j + \frac{\Delta x}{2}\left(\frac{\partial f}{\partial x}\right)_{x_j} + O\left(\Delta x^2\right)$$

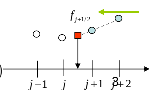


For right-wind (c<0)

$$F^{(R)}_{j+1/2} = c\left(f_{j+1} - \frac{f_{j+2} - f_{j+1}}{2}\right) = c\frac{3f_{j+1} - f_{j+2}}{2}$$

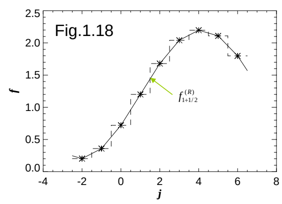
Extrapolation j+1 & j+2

$$f^{(R)}_{j+1/2} = \frac{3f_{j+1} - f_{j+2}}{2} = \frac{3f_{j+1} - \left[f_{j+1} + \Delta x\left(\frac{\partial f}{\partial x}\right)_{x_{j+1}} + O\left(\Delta x^2\right)\right]}{2} = f_{j+1} - \frac{\Delta x}{2}\left(\frac{\partial f}{\partial x}\right)_{x_{j+1}} + O\left(\Delta x^2\right)$$



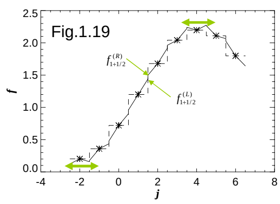
linear interpolation

$$f_{j+1/2} = \frac{f_j + f_{j+1}}{2}$$



upwind 2nd-order extrapolation

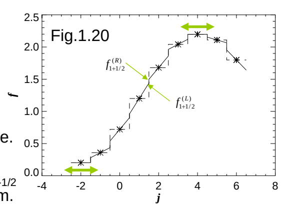
$$f^{(L)}_{j+1/2} = \frac{3f_j - f_{j-1}}{2} \qquad f^{(R)}_{j+1/2} = \frac{3f_{j+1} - f_{j+2}}{2}$$



New maxima/minima form by the extrapolation.

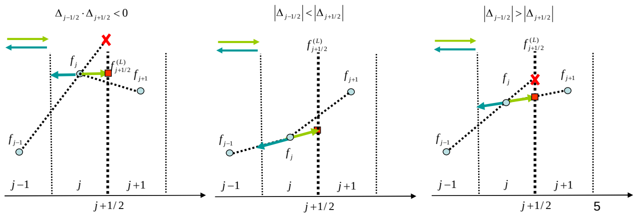
To avoid new maxima/minima

$$f^{(L)}_{j+1/2} = \begin{cases} f_j & \left(\Delta_{j-1/2}\cdot\Delta_{j+1/2} \le 0\right) \\ f_j + \frac{\Delta_{j-1/2}}{2}\min\left(1, \left|\frac{\Delta_{j+1/2}}{\Delta_{j-1/2}}\right|\right) & \left(\Delta_{j-1/2}\cdot\Delta_{j+1/2} > 0\right) \end{cases}$$



(1) At a peak, use first-order upwind scheme. f=const in a cell.

(2) Interpolate using a shallower slope in $\Delta_{j-1/2}$ and $\Delta_{j+1/2}$ to avoid a new maximum/minimum.



Godunov’s Theorem

- No second- or higher-order schemes expressed such as $f_{j,n+1} = \sum_k B_k f_{j+k,n}$ achieve monotonicity of the solution.

Central difference (second-order)

$$\frac{f_{j,n+1} - f_{j,n}}{\Delta t} + c\frac{f_{j+1,n} - f_{j-1,n}}{2\Delta x} = 0 \quad \Rightarrow \quad f_{j,n+1} = \frac{c\Delta t}{2\Delta x} f_{j-1,n} + f_{j,n} - \frac{c\Delta t}{2\Delta x} f_{j+1,n}$$

No monotonicity

$$B_{-1} = \frac{c\Delta t}{2\Delta x} \quad B_0 = 1 \quad B_1 = -\frac{c\Delta t}{2\Delta x} \quad \text{others} = 0$$

Backward difference (first-order)

$$\frac{f_{j,n+1} - f_{j,n}}{\Delta t} + c\frac{f_{j,n} - f_{j-1,n}}{\Delta x} = 0 \quad \Rightarrow \quad f_{j,n+1} = \frac{c\Delta t}{\Delta x} f_{j-1,n} + \left(1 - \frac{c\Delta t}{\Delta x}\right) f_{j,n}$$

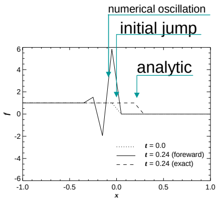
Monotonicity is possible

$$B_{-1} = \frac{c\Delta t}{\Delta x} \quad B_0 = 1 - \frac{c\Delta t}{\Delta x} \quad \text{others} = 0$$

Monotonicity vs higher-order accuracy

- We need higher-order accuracy but this leads numerical instability.
- Numerical oscillation appears near the point where *f* changes drastically.
- Strategy:
  - use higher-order difference in smooth region
  - Reduce the scheme to the first-order in a region that contains a discontinuity such as shock front.

Central difference



To construct upwind scheme with higher-order accuracy

Using $f_{j+1/2}$ extrapolated from the left $f^{(L)}_{j+1/2}$ and right $f^{(R)}_{j+1/2}$ numerical flux $F_{j+1/2}$ is constructed.

Which flux should be used $cf^{(L)}_{j+1/2}$ $(c>0)$ or $cf^{(R)}_{j+1/2}$ $(c<0)$?

$$F_{j+1/2} = \frac{cf^{(R)}_{j+1/2} + cf^{(L)}_{j+1/2}}{2} - \frac{|c|}{2}\left(f^{(R)}_{j+1/2} - f^{(L)}_{j+1/2}\right)$$ 2nd order + 1st order

$$\text{cf. } F_{j+1/2} = \frac{cf_{j+1} + cf_j}{2} - \frac{|c|}{2}\left(f_{j+1} - f_j\right)$$ 1st order

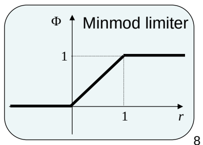
with

$$f^{(L)}_{j+1/2} = f_j + \frac{\Delta_{j-1/2}}{2}\Phi\left(\frac{\Delta_{j+1/2}}{\Delta_{j-1/2}}\right)$$

$$f^{(R)}_{j-1/2} = f_j - \frac{\Delta_{j+1/2}}{2}\Phi\left(\frac{\Delta_{j-1/2}}{\Delta_{j+1/2}}\right)$$

limiter

$$\Phi(r) = \begin{cases} 0 & (r \le 0) \\ r & (0 < r \le 1) \\ 1 & (r > 1) \end{cases}$$



To construct a scheme with a higher-order accuracy in time

1st-order accuracy in time

$f_{j,n}$

↓ Monotonicity preserving extrapolation

$f^{(R)}_{j+1/2,n}$ $f^{(L)}_{j+1/2,n}$

↓ Upwind numerical flux at t

$F_{j+1/2,n}$

↓ Time advance Δt using the above flux

$$f_{j,n+1} = f_{j,n} - \frac{\Delta t}{\Delta x}\left(F_{j+1/2,n} - F_{j-1/2,n}\right)$$

2nd-order accuracy in time

$f_{j,n}$

↓

$f^{(R)}_{j+1/2,n}$ $f^{(L)}_{j+1/2,n}$

↓ Upwind numerical flux at *t*

$F_{j+1/2,n}$

↓ Get *f* at *t*+Δ*t*/2 using the flux at *t*

$$f_{j,n+1/2} = f_{j,n} - \frac{\Delta t}{2\Delta x}\left(F_{j+1/2,n} - F_{j-1/2,n}\right)$$

↓

$f^{(R)}_{j+1/2,n+1/2}$ $f^{(L)}_{j+1/2,n+1/2}$

↓ Upwind numerical flux at *t*+Δ*t*/2

$F_{j+1/2,n+1/2}$

↓ Time advance Δ*t* using the flux at *t*+Δ*t*

$$f_{j,n+1} = f_{j,n} - \frac{\Delta t}{\Delta x}\left(F_{j+1/2,n+1/2} - F_{j-1/2,n+1/2}\right)$$